Multivariable Control (ME-422) - Exercise session 2

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1. **Exercise on modal analysis**
   Consider the system $x^+ = Ax$ where

$$A = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -0.5 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Tell if

   (a) all modes go to zero as $k \to \infty$
   (b) all modes are bounded
   (c) there is at least an unbounded mode.

   Hint: Note that A is block diagonal and its eigenvalues are those of each individual block.

2. **Complex conjugate eigenvalues generate real modes**
   At first sight, it might seem strange that complex conjugate eigenvalues give rise to modes taking values in $\mathbb{R}$ instead of $\mathbb{C}$. This exercise provides some intuition on why this happens.

   Consider the system

$$\begin{cases} x^+ = Ax \\ y = Cx \end{cases}$$
$$x(0) = x_0.$$

   (a) Let $A \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{C}^{n\times n}$ be nonsingular matrices.
   Define $\hat{x}(k) = Tx(k) \in \mathbb{C}^n$. Show that $\hat{x}(k)$ follows the dynamics $\hat{x}^+ = \hat{A}\hat{x}$ with $\hat{A} = TAT^{-1}$.
   (b) Consider the case

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

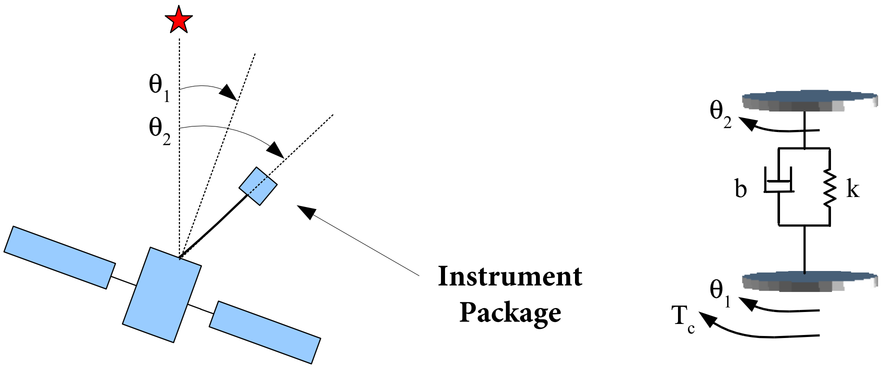
   where the eigenvalues of A are $1 \pm j = \sqrt{2}e^{\pm j\frac{\pi}{4}}$ and

$$\hat{A} = TAT^{-1} = \begin{bmatrix} 1+j & 0 \\ 0 & 1-j \end{bmatrix} \text{ for } T^{-1} = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}.$$

   Compute explicitly $x(k)$ and $y(k)$ as a function of x_0.

3. **Satellite simulation**
   Consider the satellite in the figure. The satellite mission requires accurate pointing of a scientific sensor package with respect to a fixed star.

   - θ_2 is the angle deviation of the instrument package from the reference star. It can be measured through devices called "star trackers."



- θ_1 is the angle deviation of the satellite body

By using cold-gas jets, one can produce a torque T_c acting on the satellite body.

The figure also shows an equivalent mechanical model of the satellite. This system is composed by two rotating masses connected by a spring with torque constant κ and viscous-damping constant b.
A discrete-time model of the system with state $x = \begin{bmatrix}\theta_2 & \dot{\theta}_2 & \theta_1 & \dot{\theta}_1\end{bmatrix}$, output $y = \theta_2$, and input $u = T_c$ can be written as

$$\begin{cases} x^+ = \begin{bmatrix} 1 & T & 0 & 0 \\ -T\frac{\kappa}{J_2} & 1 - T\frac{b}{J_2} & T\frac{\kappa}{J_2} & T\frac{b}{J_2} \\ 0 & 0 & 1 & T \\ T\frac{\kappa}{J_1} & T\frac{b}{J_1} & -T\frac{\kappa}{J_1} & 1 - T\frac{b}{J_1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ T\frac{1}{J_1} \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x, \end{cases}$$

where $T > 0$ is the sampling time and J_1 and J_2 are the inertias of the masses. Set $\kappa = 0.091$, $b = 0.0036$, $J_1 = 1$, $J_2 = 0.1$, and $T = 0.01$.

(a) Check if the system has unbounded modes. Use MATLAB for computations.

(b) Draw the free output between 0 and $200s$., starting from the initial condition $x = [0 \quad 0 \quad 0.1 \quad 0]^T$. Give a physical interpretation to the result.

(c) Draw the impulse response (that is, the output generated by the input $u(0) = 1,\ u(k) = 0,\ k > 0$) and give an interpretation of the results in terms of the satellite motion.

(d) Now consider the output as $y = x_2$. Draw the impulse response and, through the physical interpretation of the results, explain why its asymptotic value is consistent with the graph obtained in the previous point.

Useful MATLAB commands: `eig`, `null`, `ss(A,B,C,D,T)`, `lsim`, `impulse`.