Multivariable Control (ME-422) - Exercise session 12

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1. Consider the system

$$\begin{aligned} x_{k+1} &= 0.5x_k + w_k \qquad w_k \sim N(0,1) \\ y_k &= x_k + v_k \qquad v_k \sim N(0,1) \\ x_0 &\sim N(0,1) \end{aligned}$$

   and let the usual statistical assumptions for KF hold.

   Derive a nonrecursive expression of $\mathrm{E}[x_1|y_0, y_1]$.

2. Consider the first-order system

$$\begin{aligned} x_{k+1} &= \alpha x_k + w_k \\ y_k &= \gamma x_k + v_k \\ x_0 &\sim N\left(\bar{x}_0, \bar{\Sigma}_0\right) \end{aligned} \tag{1}$$

   where α and γ are parameters, $w_k \sim N(0, \beta^2)$, $v_k \sim N(0,1)$, and the usual statistical assumptions for Kalman Filtering hold (see the lectures).

   We will study the Difference Riccati Equation (DRE) associated to the Kalman Predictor.

   (a) Assume the system is unstable, that is $\alpha > 1$, and that $\beta \neq 0$, $\gamma \neq 0$. We want to check if the Kalman Predictor can "track" the state even when it diverges.

      i. Show that, using Σ_k for $\Sigma_{k|k-1}$, the DRE is

$$\Sigma_{k+1} = \beta^2 + \frac{\alpha^2 \Sigma_k}{1 + \gamma^2 \Sigma_k}. \tag{2}$$

      ii. The values of Σ_k can be computed from (2) using the following graphical procedure (valid also for a generic nonlinear system $x_{k+1} = f(x_k)$).

         A. Plot $f(\Sigma_k) = \beta^2 + \frac{\alpha^2 \Sigma_k}{1+\gamma^2 \Sigma_k}$ and the line $l(\Sigma_k) = \Sigma_k$. The intersections of f and l gives the solutions to the ARE

$$\bar{\Sigma} = \beta^2 + \frac{\alpha^2 \bar{\Sigma}}{1 + \gamma^2 \bar{\Sigma}}. \tag{3}$$

         This plot is given in Figure 1. Consider only the positive solution.

         B. Set $k = 0$ and fix $\Sigma_0 > 0$ on the horizontal axis. One has $\Sigma_{k+1} = f(\Sigma_k)$ and
            - If $\Sigma_k < \bar{\Sigma}$, then $\Sigma_{k+1} > \Sigma_k$ (because $f(\Sigma_k) > l(\Sigma_k)$) and Σ_k increases monotonically towards $\bar{\Sigma}$
            - If $\Sigma_k > \bar{\Sigma}$, then $\Sigma_{k+1} < \Sigma_k$ (because $f(\Sigma_k) < l(\Sigma_k)$) and Σ_k decreases monotonically towards $\bar{\Sigma}$

         Plot on the figure, the (qualitative) sequences Σ_0, Σ_1, Σ_2, ... when starting from $\Sigma_0 > \bar{\Sigma}$ and $\Sigma_0 < \bar{\Sigma}$.

   (b) Assume now that $|\alpha| > 1$, $\beta \neq 0$ but $\gamma = 0$.

      i. Derive the DRE.

      ii. Adapt the graphical method in point (2.a.ii.) for assessing if the DRE is converging or not.

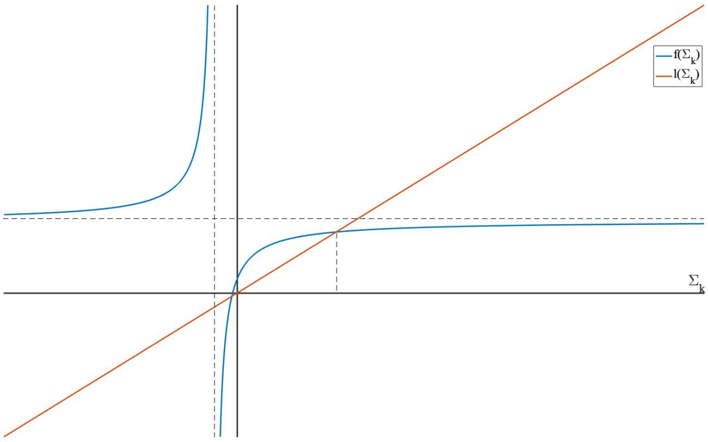

Figure 1: $f(\Sigma_k)$ and $l(\Sigma_k)$

(c) Assume that $|\alpha| > 1$, $\beta^2 = 0$, $\gamma \neq 0$. Use MATLAB for plotting the new functions $f(\Sigma_k)$ and $l(\Sigma_k)$ and repeat the analysis about the convergence of the DRE.

(d) The previous analysis shows that Σ_k diverges when $\gamma = 0$. Can you provide an interpretation of this result (**Hint:** Look at (1))?