Multivariable Control (ME-422) - Exercise session 11 SOLUTIONS

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1. Let v and w be jointly Gaussian, independent, with zero average and $var[v] = 2$, $var[w] = 1$. Let x and y be defined in the following alternative ways

$$(a)\quad \begin{aligned} x &= v + w \\ y &= v - w \end{aligned} \qquad (b)\quad \begin{aligned} x &= 2v + 2w \\ y &= 2v - 2w \end{aligned} \qquad (c)\quad \begin{aligned} x &= v + w \\ y &= 2v + 2w \end{aligned} \qquad (d)\quad \begin{aligned} x &= v \\ y &= \sqrt{2}w \end{aligned}$$

Which pairs x, y have the highest/lowest covariance?

Solution: In all cases, there is $A_i \in \mathbb{R}^{2\times 2}, i = 1, \dots, 4$ such that $\begin{bmatrix} x \\ y \end{bmatrix} = A_i \begin{bmatrix} v \\ w \end{bmatrix}$. Then we have that

$$\mathrm{E}\left[\begin{bmatrix} x \\ y \end{bmatrix}\right] = A_i \mathrm{E}\left[\begin{bmatrix} v \\ w \end{bmatrix}\right] = 0.$$

Moreover, it is given that $V = var\left(\begin{bmatrix} v \\ w \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Recall that $cov[x, y] = \mathrm{E}\left[(x - \mathrm{E}[x])(y - \mathrm{E}[y])\right]$.

(a)

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad V_1 = var\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A_1 V A_1^T = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Hence, $cov[x, y] = 1$ in this case.

(b)

$$A_2 = 2A_1 \to V_2 = var\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 4V_1$$

Hence, $cov[x, y] = 4$ in this case.

(c)

$$A_3 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \qquad V_1 = var\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A_3 V A_3^T = \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}.$$

Hence, $cov[x, y] = 6$ in this case.

(d)

$$A_4 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix} \qquad V_1 = var\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A_4 V A_4^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Hence, $cov[x, y] = 0$ in this case.

Conclusion: Case (c) produces the highest covariance, whereas case (d) produces the lowest.

2. Consider the following data

$$\begin{aligned} u_1 &= -1 \qquad & u_2 &= \sqrt{0.5} \qquad & u_3 &= 1 \\ y_1 &= 0 \qquad & y_2 &= 0 \qquad & y_3 &= 1.5 \end{aligned}$$

and the following model

$$y_k = \theta u_k^2 + v_k$$

where
$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, V\right) \qquad V = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover, $\theta \sim N(1,1)$ and is independent of $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$.

Compute the parameter estimate $\hat{\theta} = \mathrm{E}[\theta | y_1, y_2, y_3]$.

Hint: Derive first the probability density of $\begin{bmatrix} \theta & y_1 & y_2 & y_3 \end{bmatrix}$. Use MATLAB for computing the required matrix products.

Solution: Using the data model we have

$$\underbrace{\begin{bmatrix} \theta \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\xi} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0.5 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \theta \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\psi}$$

Since ψ is Gaussian with

$$\mathrm{E}[\psi] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$var[\psi] = \Psi = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & V \end{array}\right]$$

then, as seen in the lectures, ξ is also Gaussian with

$$\mathrm{E}[\xi] = A\mathrm{E}[\psi] = \begin{bmatrix} 1 \\ 1 \\ 0.5 \\ 1 \end{bmatrix}$$

$$var[\xi] = A\Psi A^T = \left[\begin{array}{c|c} C_{\theta\theta} & C_{\theta Y} \\ \hline C_{Y\theta} & C_{YY} \end{array}\right] = \left[\begin{array}{c|ccc} 1 & 1 & 0.5 & 1 \\ \hline 1 & 2 & 1 & 1 \\ 0.5 & 1 & 1.25 & 0.5 \\ 1 & 1 & 0.5 & 2 \end{array}\right]$$

Then, setting $Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$, we get

$$\begin{aligned} \mathrm{E}[\theta|Y] &= \mathrm{E}[\theta] + C_{\theta Y} C_{YY}^{-1} (Y - \mathrm{E}[Y]) \\ &= 1 + \begin{bmatrix} 1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1.25 & 0.5 \\ 1 & 0.5 & 2 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ 0 \\ 1.5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix} \right) \\ &= 0.8333 \end{aligned}$$

3. Find the update rule for the covariance matrix $P_k = \mathrm{E}\left[(x_k - \mathrm{E}[x_k])(x_k - \mathrm{E}[x_k])^T\right]$ of the process

$$x_{k+1} = Ax_k + Bw_k \quad A = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$x_0 \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

where $w_k \sim WGN(0,1)$. Is P_k convergent as $k \to \infty$? If yes, compute the limit value.

Solution: As seen in the lectures

$$P_{k+1} = AP_kA^T + W$$

where $W = B1B^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

A is Hurwitz (both eigenvalues have modulus 0.5) and hence, $P_k \to \bar{P}$ where $\bar{P} = \bar{P}^T$ verifies

$$\bar{P} = A\bar{P}A^T + W. \tag{1}$$

Let $\bar{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$. Then, (1) becomes

$$\begin{aligned}
\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} &= \frac{1}{4}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} P_{12} & -P_{11}+P_{12} \\ P_{22} & -P_{12}+P_{22} \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} P_{22}+4 & P_{22}-P_{12}+4 \\ P_{22}-P_{12}+4 & P_{11}-2P_{12}+P_{22}+4 \end{bmatrix}
\end{aligned}$$

The solution to the system

$$\begin{cases} P_{11} = \dfrac{P_{22}}{4} + 1 \\ P_{12} = \dfrac{1}{4}(P_{22} - P_{12}) + 1 \\ P_{22} = \dfrac{1}{4}(P_{11} - 2P_{12} + P_{22}) + 1 \end{cases}$$

is $P_{11} = 1.2698$, $P_{12} = 1.0159$, $P_{22} = 1.0794$. One can verify that $\bar{P} > 0$ as its eigenvalues are 0.1543 and 2.1949.

4. Recall the eigenvalue assignment theorem

Theorem. For a given pair $(A, B), A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, $\exists K \in \mathbb{R}^{m\times n}$:
$(A + BK)$ has prescribed eigenvalues $\iff$ (A, B) is reachable.

Prove the $\implies$ statement.

Hints: In an equivalent way, one can try to prove
(A, B) unreachable $\implies$ not all eigenvalues of $(A + BK)$ can be assigned.
Use the reachability form of (A, B) for showing the implication.

Solution: The reachability form of (A, B) is

$$\hat{A} = \begin{bmatrix} \hat{A}_a & \hat{A}_{ab} \\ 0 & \hat{A}_b \end{bmatrix} \qquad \hat{B} = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix}$$

where $\hat{A}_b \in \mathbb{R}^{n_0\times n_0}$, $n_0 \geq 1$. Using the controller $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ $K_2 \in \mathbb{R}^{n_0\times m}$, one obtains

$$\hat{A} + \hat{B}K = \begin{bmatrix} \hat{A}_a + \bar{B}K_1 & \hat{A}_{ab} + \bar{B}K_2 \\ 0 & \hat{A}_b \end{bmatrix}$$

which is block-triangular. Hence, the eigenvalues of $(\hat{A} + \hat{B}K)$ are the union of the eigenvalues of $(\hat{A}_a + \bar{B}K_1)$ and $\hat{A}_b$. Since the eigenvalues of $\hat{A}_b$ can not be modified by the control gain K, not all eigenvalues of $(\hat{A} + \hat{B}K)$ can be set to desired locations.