Multivariable Control (ME-422) - Exercise session 11

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1. Let v and w be jointly Gaussian, independent, with zero average and $var[v] = 2$, $var[w] = 1$. Let x and y be defined in the following alternative ways

$$(a)\quad \begin{aligned} x &= v + w \\ y &= v - w \end{aligned} \qquad (b)\quad \begin{aligned} x &= 2v + 2w \\ y &= 2v - 2w \end{aligned} \qquad (c)\quad \begin{aligned} x &= v + w \\ y &= 2v + 2w \end{aligned} \qquad (d)\quad \begin{aligned} x &= v \\ y &= \sqrt{2}w \end{aligned}$$

   Which pairs x, y have the highest/lowest covariance?

2. Consider the following data

$$\begin{aligned} u_1 &= -1 \qquad & u_2 &= \sqrt{0.5} \qquad & u_3 &= 1 \\ y_1 &= 0 & y_2 &= 0 & y_3 &= 1.5 \end{aligned}$$

   and the following model

$$y_k = \theta u_k^2 + v_k$$

   where

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, V \right) \qquad V = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Moreover, $\theta \sim N(1, 1)$ and is independent of $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$.

   Compute the parameter estimate $\hat{\theta} = \mathrm{E}\left[\theta | y_1, y_2, y_3\right]$.

   Hint: Derive first the probability density of $\begin{bmatrix} \theta & y_1 & y_2 & y_3 \end{bmatrix}$. Use MATLAB for computing the required matrix products.

3. Find the update rule for the covariance matrix $P_k = \mathrm{E}\left[(x_k - \mathrm{E}[x_k]) (x_k - \mathrm{E}[x_k])^T \right]$ of the process

$$\begin{aligned} x_{k+1} &= A x_k + B w_k \quad A = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ x_0 &\sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \end{aligned}$$

   where $w_k \sim WGN(0, 1)$. Is P_k convergent as $k \to \infty$? If yes, compute the limit value.

4. Recall the eigenvalue assignment theorem

   Theorem. For a given pair $(A, B), A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$, $\exists K \in \mathbb{R}^{m \times n}$:
   $(A + BK)$ has prescribed eigenvalues $\iff$ (A, B) is reachable.

   Prove the $\implies$ statement.

   Hints: In an equivalent way, one can try to prove
   (A, B) unreachable $\implies$ not all eigenvalues of $(A + BK)$ can be assigned.
   Use the reachability form of (A, B) for showing the implication.