Exercise 1A

Stability of an equilibrium and modal analysis

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LTI system: stability of equilibria

Let $(\bar{x}, \bar{u})$ be an equilibrium for $x^+ = Ax + Bu$, $x(0) = x_0$. How uncertainty $x_0 = \bar{x}$ propagates to $x(k)$?

- Perturbed experiment : $\tilde{x}(k) = \phi(k, 0, \tilde{x}_0, \bar{u})$

Definitions (Lyapunov stability)

The equilibrium state $\bar{x}$ is

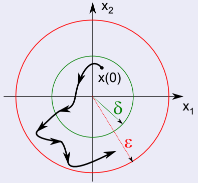
- stable if $\forall \epsilon > 0\ \exists \delta > 0 : \|\tilde{x}_0 - \bar{x}\| \leq \delta \Rightarrow \|\tilde{x}(k) - \bar{x}\| < \epsilon, \forall k \geq 0$
- (globally) asymptotically stable (AS) if it is stable and attractive, i.e.,

$$\lim_{k \to \infty} \|\tilde{x}(k) - \bar{x}\| = 0,\ \forall \tilde{x}_0 \in \mathbb{R}^n$$

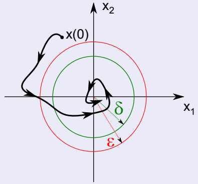
- unstable if not stable

LTI system: stability of equilibria

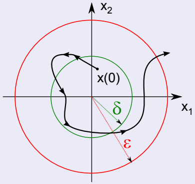
$\bar{x} = 0$ stable



$\bar{x} = 0$ AS



$\bar{x} = 0$ unstable



Definition

$\bar{x}$ is (globally) **exponentially stable** (ES) if there are $\alpha > 0, \rho \in [0, 1)$ such that

$$\|\tilde{x}(k) - \bar{x}\| \leq \alpha\rho^k \|\tilde{x}_0 - \bar{x}\|, \quad \forall \tilde{x}_0 \in \mathbb{R}^n$$

where the constant β such that $\rho = \mathrm{e}^{\beta}$ is the **decay rate**.

↳ $\rho^k = e^{\beta k}$

Problem

Definitions are difficult to use. How to test stability?

- Before addressing stability analysis, we need a number of tools (equivalent systems and modes)

Review: equivalent LTI systems

$$x^+ = Ax + Bu$$
$$y = Cx + Du$$

- Change of coordinates $\hat{x}(k) = Tx(k)$, $T \in \mathbb{R}^{n \times n}$ invertible.

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$T^{-1}\hat{x}(k)$

$$\begin{aligned} \hat{x}(k+1) &= Tx(k+1) = T(Ax(k) + Bu(k)) = T(AT^{-1}\hat{x}(k) + Bu(k)) \\ &= TAT^{-1}\hat{x}(k) + TBu(k) = \hat{A}\hat{x}(k) + \hat{B}u(k) \\ & \hat{A} = TAT^{-1}, \quad \hat{B} = TB \end{aligned}$$

Review: equivalent LTI systems

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\begin{aligned}
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y &= Cx + Du
\end{aligned}
$$

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&= TAT^{-1}\hat{x}(k) + TBu(k) = \hat{A}\hat{x}(k) + \hat{B}u(k) \\
&\quad \hat{A} = TAT^{-1}, \quad \hat{B} = TB \\
y(k) &= Cx(k) + Du(k) = CT^{-1}\hat{x}(k) + Du(k) = \hat{C}\hat{x}(k) + \hat{D}u(k) \\
&\quad \hat{C} = CT^{-1}, \quad \hat{D} = D
\end{aligned}
$$

Review: equivalent LTI systems

$$x^+ = Ax + Bu$$
$$y = Cx + Du$$

$$\hat{x}^+ = \hat{A}\hat{x} + \hat{B}u$$
$$y = \hat{C}\hat{x} + \hat{D}u$$

Definition

The system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is *equivalent* to the system (A, B, C, D) in the sense that for an input $u(k)$, $k \geq 0$ and two initial states x_0 e $\hat{x}_0$ verifying $\hat{x}_0 = Tx_0$, the state trajectories verify $\hat{x}(k) = Tx(k)$, $k \geq 0$, and outputs are identical

Remark

A and $\hat{A}$ are similar $\Rightarrow$ they have the same eigenvalues

Analysis of the *free* state

$$x(k+1) = Ax(k), \quad x(0) = x_0 \implies x(k) = A^k x_0$$

Theorem

Each scalar entry of the matrix A^k is a linear combination of functions of time, called **modes**, *associated to* **distinct** *eigenvalues of A as follows*

eigenvalues	*modes*
$\lambda_i \in \mathbb{R}$	$\begin{cases} 0 & \text{for } k < p_i \\ k^{p_i}\lambda_i^{k-p_i} & \text{for } k \geq p_i \end{cases}, \quad p_i = 0, 1, \ldots, \eta_i - 1$ ↳ $p_i = 0$ λ_i^k, $k \geq 0$
$\lambda_i = \rho_i e^{j\theta_i}$ *AND* $\lambda_h = \lambda_i^*$	$\begin{cases} 0 & \text{for } k < p_i \\ k^{p_i}\rho_i^{k-p_i}\sin(\theta_i(k-p_i)+\varphi_i) & \text{for } k \geq p_i \end{cases}$ $p_i = 0, 1, \ldots, \eta_i - 1$ ↳ $p_i = 0$ $\rho_i^k \sin(\theta_i k + \varphi_i)$

where

- η_i *is a suitable integer verifying*

$$1 \leq \eta_i \leq n_i$$
$$\eta_i = 1 \iff n_i = \nu_i$$

- n_i *and* ν_i *are, respectively, the algebraic and geometric multiplicity of the eigenvalue* λ_i
- $\varphi_i \in \mathbb{R}$ *is a suitable parameter*

Recall

- n_i: how many times λ_i is a root of the characteristic polynomial of A
- $\nu_i = \dim(V_{\lambda_i}),\ V_{\lambda_i} = \{v_i \in \mathbb{R}^n |\, Av_i = \lambda_i v_i\}$ = eigenspace associated to λ_i
- $\nu_i \leq n_i$, always!

Remarks on the theorem

- partial characterisation of modes, because (i) the value of η_i is not precisely given if $n_i \neq \nu_i$ and (ii) $\varphi_i \in \mathbb{R}$ is not given
  $\implies$ one can show that η_i is the dimension of the largest Jordan block associated to λ_i
- If $\eta_i = 1$:
  - a single mode associated to a real λ_i
  - a single mode associated to a complex conjugate pair $\lambda_i = \lambda_h^*$
    Sanity check: 2 degrees of freedom defining the pair of eigenvalues
    $\implies$ 2 degrees of freedom θ_i and ρ_i defining the mode.

Important remark

Free states $x(k) = A^k x_0$ are linear combinations of the modes (through x_0)

eigenvalues	modes
$\lambda_i \in \mathbb{R}$	$\begin{cases} 0 & \text{for } k < p_i \\ k^{p_i}\lambda_i^{k-p_i} & \text{for } k \geq p_i \end{cases}, \quad p_i = 0, 1, \ldots, \eta_i - 1$
$\lambda_i = \rho_i e^{j\theta_i}$ AND $\lambda_h = \lambda_i^*$	$\begin{cases} 0 & \text{for } k < p_i \\ k^{p_i}\rho_i^{k-p_i}\sin(\theta_i(k-p_i)+\varphi_i) & \text{for } k \geq p_i \end{cases}$ $p_i = 0, 1, \ldots, \eta_i - 1$

- λ_i simple $\implies n_i = \nu_i = 1 \implies \eta_i = 1$ and hence $p_i = 0$
- When λ_i is not simple, we are not interested in computing the precise value of η_i but only to know when $\eta_i = 1$ (hence $p_i = 0$) or $\eta_i > 1$ (hence p_i takes the values 0 and 1, at least)

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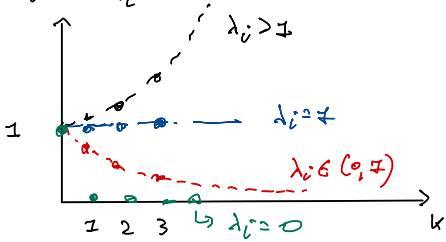
Example

Simple eigenvalue

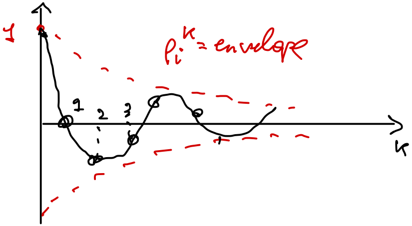
$\rightarrow \eta_i = 1 \rightarrow \nu_i = 1 \rightarrow \eta_i = 1 \rightarrow p_i = 0$

$\lambda_i \geq 0 \rightarrow \lambda_i^k$



$$\begin{cases} \lambda_i = \rho_i e^{j\vartheta_i} \\ \lambda_h = \lambda_i^* \end{cases} \rightarrow \rho_i^k \sin(\vartheta_i k + \varphi_i)$$

If $\rho_i \in [0, 1)$



Example

Compute all modes associated to A

- $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \lambda_1 = 2$. Diagonalisable! $\implies n_1 = \nu_1 = 2$
  $\rightarrow$ one mode 2^k
- $A = \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 0.5 & 1 \\ 0 & 0 & 0.5 \end{bmatrix}, \lambda_1 = 0.5, n_1 = 3, \nu_1 = ?$

$$V_{0.5} = \{v : (A - 0.5I)v = 0\} = \left\{ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} : v_2 = v_3 = 0 \right\}$$

$$= \left\{ v = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R} \right\} \implies \dim(V_{0.5}) = 1$$

From the table

eigenvalues	modes
$\lambda_i \in \mathbb{R}$	$\begin{cases} 0 & \text{for } k < p_i \\ k^{p_i}\lambda_i^{k-p_i} & \text{for } k \geq p_i \end{cases}, \quad p_i = 0, 1, \ldots, \eta_i - 1$
$\lambda_i = \rho_i e^{j\theta_i}$ AND $\lambda_h = \lambda_i^*$	$\begin{cases} 0 & \text{for } k < p_i \\ k^{p_i}\rho_i^{k-p_i}\sin(\theta_i(k-p_i)+\varphi_i) & \text{for } k \geq p_i \end{cases}$ $p_i = 0, 1, \ldots, \eta_i - 1$

- $\underbrace{0.5^k}_{p_i = 0}$ and $\overbrace{\begin{cases} 0 & \text{for } k = 0 \\ k\,0.5^{k-1} & \text{for } k > 0 \end{cases}}^{p_i = 1}$ are modes
- One could also have the mode $\begin{cases} 0 & \text{for } k = 0, 1 \\ k^2\,0.5^{k-2} & \text{for } k > 1 \end{cases}$ if $\eta_i = 3$, but we need a deeper analysis for assessing whether this is true

- $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \varphi(\lambda) = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}\right) = \lambda^2 + 1,$
  setting $\varphi(\lambda) = 0$ we get

$$\lambda^2 = -1 \implies \begin{cases} \lambda_1 = j = 1\mathrm{e}^{j\frac{\pi}{2}} \implies n_1 = 1 \implies \nu_1 = 1 & p_1 = 0 \\ \lambda_2 = -j = 1\mathrm{e}^{-j\frac{\pi}{2}} \implies n_2 = 1 \implies \nu_2 = 1 & p_2 = 0 \end{cases}$$

eigenvalues	modes
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- Just one mode associated to the pair λ_1, λ_2:

$$1^k \sin\left(\frac{\pi}{2}k + \varphi_i\right)$$

Macroscopic behaviour of modes

Lemma

- If $|\lambda_i| < 1$, all modes associated to λ_i are **bounded** and **go to zero** as $k \to +\infty$.
- If $|\lambda_i| > 1$, all modes associated to λ_i are **unbounded**.
- If $|\lambda_i| = 1$ and $\nu_i = n_i$, all modes associated to λ_i are **bounded**.
- If $|\lambda_i| = 1$ and $\nu_i < n_i$, there's an **unbounded** mode associated to λ_i

Proof

Follows from the table of the modes:

eigenvalues	modes
$\lambda_i \in \mathbb{R}$	$\begin{cases} 0 & \text{for } k < p_i \\ k^{p_i}\lambda_i^{k-p_i} & \text{for } k \geq p_i \end{cases}, \quad p_i = 0, 1, \dots, \eta_i - 1$
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System with simple eigenvalues and modulus > 1

(simple $\rightarrow p_i = 0$)

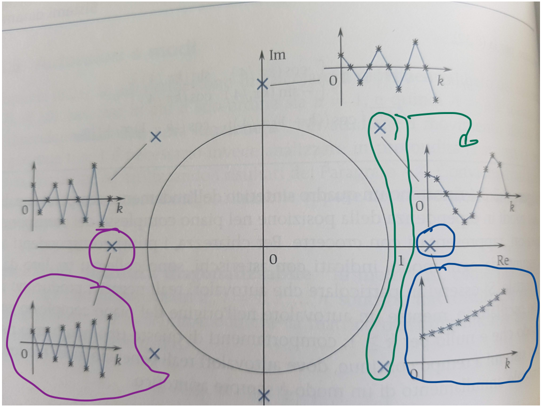

Figure: [P. Bolzern, R. Scattolini, N. Schiavoni, *Fondamenti di controlli automatici*, 4th edition, McGraw Hill Education, 2015]

eigenvalues	modes
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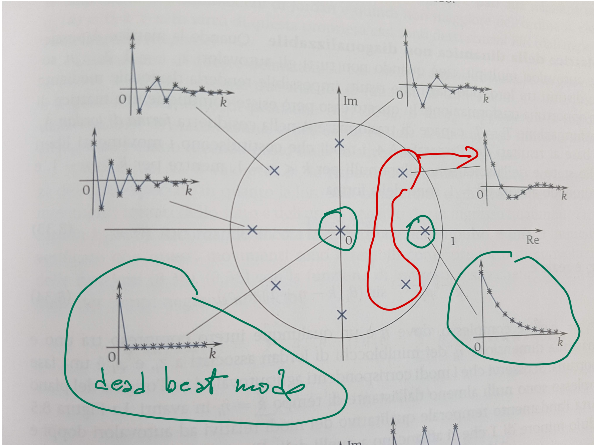
System with simple eigenvalues and modulus < 1

(simple $\rightarrow p_i = 0$)



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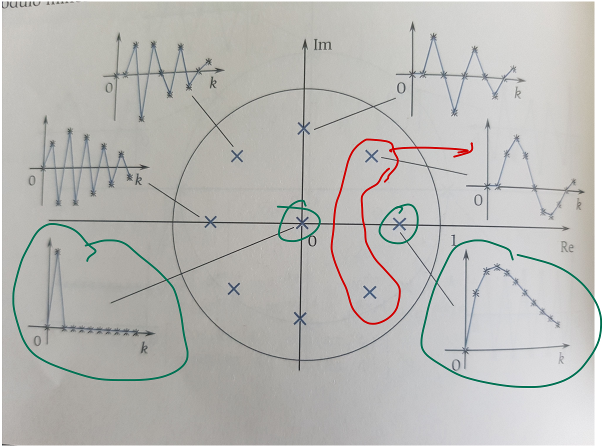
System with double eigenvalues of modulus < 1, positioned as in the previous figure, and with $\eta_i = 2$

Additional modes corresponding to $p_i = 1$



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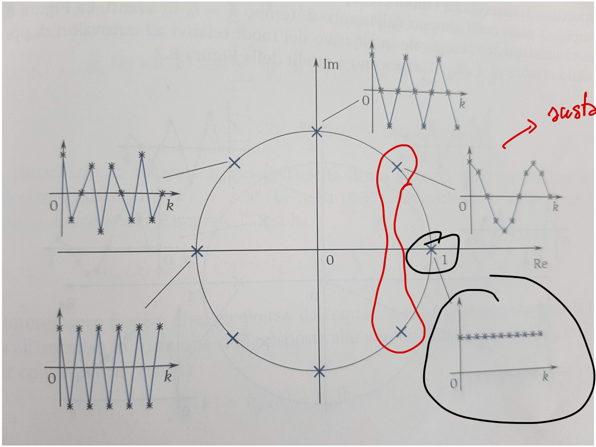
System with simple eigenvalues and modulus = 1

↳ $p_i = 0$ → bounded modes



sustained oscillations

eigenvalues	modes
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Take-home messages

- Stability of an equilibrium: key property but challenging to verify by using definitions
- modes = key functions for analysing LTI systems
  - associated to eigenvalues
  - finitely many

Next: how to use modes to characterise stability of LTI systems