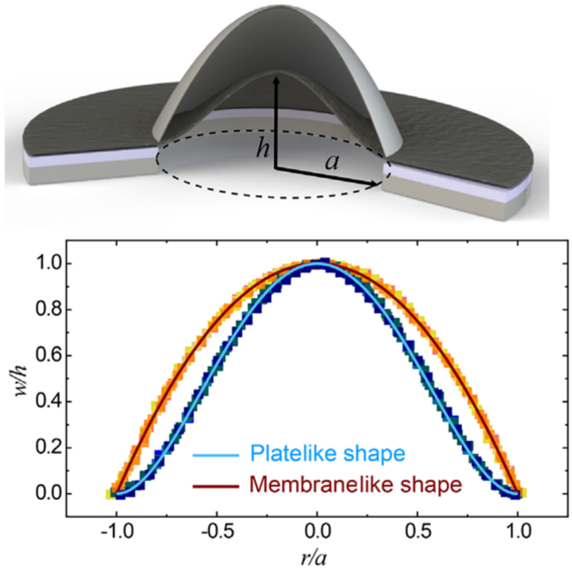
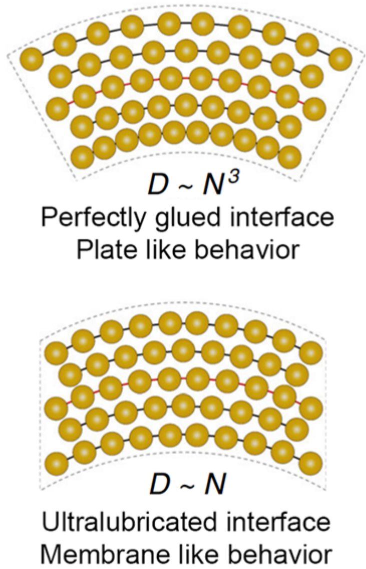
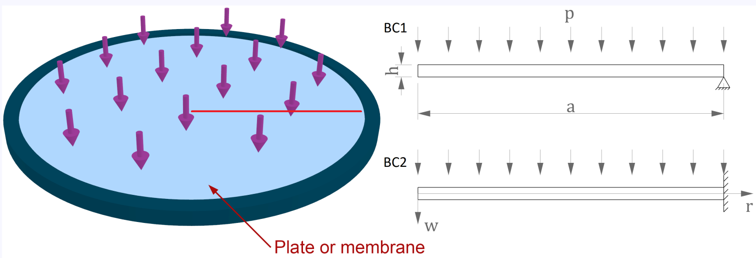




Studio 8: Circular plate under distributed load

Exercise 8.1 In this studio, we will study two different problems that can be solved using limiting cases of the Föppl–von Kármán equations. Wang *et al.*, *Phys. Rev. Lett.* (2019) have shown that multiple layers of graphene undergoing bending can exhibit a plate-like response when the strength of inter-layer attractions is large (so minimal inter-layer shearing), but a membrane-like response for small inter-layer attractions (significant shearing). Consider a circular plate (radius a, thickness h) that is supported along its edge and loaded with a uniform downward pressure p (dead load), as shown below. We will investigate its displacement behavior in these two scenarios.



S 8.0.1 **Problem 1: Pure bending**

The first problem considers the behavior of a plate undergoing pure bending with the assumption that the displacements are small.

Questions:

1. How do we account for the work done by the lateral pressure p (applied in the w direction, where w is the downward displacement of the plate) in the governing equations?

2. How do we simplify the Föppl–von Kármán equations using the assumption of small displacements?

   Hint: In terms of the out-of-plane displacement w and in-plane stress tensor

$\sigma^o_{\alpha\beta}$, the Föppl–von Kármán equations, derived in the lectures, are

$$\frac{Eh^3}{12(1-\nu^2)}\Delta^2 w - h\sigma^o_{\alpha\beta}\frac{\partial^2 w}{\partial x_\alpha \partial x_\beta} = p, \tag{8.1}$$

$$\frac{\partial \sigma^o_{\alpha\beta}}{\partial x_\alpha} = 0, \tag{8.2}$$

where E is the Young's modulus, ν is the Poisson's ratio and the indices α, β vary over the in-plane directions $1, 2$. For convenience we often write

$$D := \frac{Eh^3}{12(1-\nu^2)}. \tag{8.3}$$

3. What is an appropriate coordinate system for this problem, and how do we write the simplified governing equations in such a coordinate system?

4. Considering axisymmetric deformations, what is the displacement w of the plate if its boundary is clamped horizontally?

   Hint: To obtain all relevant boundary conditions needed to solve for w, you may find it helpful to consider the moment and transverse shear force within the plate. In the lectures (section 6.2, page 84) we derived the internal moment tensor $m_{\alpha\beta}$:

$$m_{\alpha\beta} = \frac{E}{1-\nu^2}\left[(1-\nu)\mathcal{K}_{\alpha\beta} + \nu\delta_{\alpha\beta}\mathcal{K}_{\gamma\gamma}\right], \tag{8.4}$$

   where $\mathcal{K}_{\alpha\beta}$ is the curvature tensor. In the polar coordinate system, we can then calculate the principle curvatures and integrate over the thickness to obtain the radial and azimuthal moment resultant:

$$M_{rr} = \frac{h^3}{12}m_{rr} = -D\left(\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{\nu}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right), \quad M_{\theta\theta} = \frac{h^3}{12}m_{\theta\theta} = -D\left(\nu\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right). \tag{8.5}$$

   The resultant shear force (directed upwards) is

$$Q = D\left(\frac{\mathrm{d}^3 w}{\mathrm{d}r^3} + \frac{1}{r}\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} - \frac{1}{r^2}\frac{\mathrm{d}w}{\mathrm{d}r}\right). \tag{8.6}$$

5. What is the displacement of the plate if we instead assume its boundary is simply supported?

6. What is the stress distribution within the plate?

S 8.0.2 Problem 2: Pure stretching

In this second problem, we consider the behavior of a circular membrane with the assumption that there is no bending stiffness.

1. How do we simplify the Föppl–von Kármán equations to exclude the bending energy? What other equations are now needed to obtain a closed system of equations for w and the stress components $\sigma^o_{\alpha\beta}$?

2. Again assuming axisymmetric deformations, how do we write the simplified governing equations in polar coordinates?

3. If we further assume (for simplicity) that the in-plane displacement $u_r = 0$ everywhere, what is the displacement of the plate?

4. **Bonus problem:** If we relax the constraint that the in-plane displacement $u_r = 0$, what is the displacement of the plate?

S 8.1 Further hints:

Problem 1

1. Include external work into the variation of the total energy.
2. The equations should simplify to $D\Delta^2 w = p$. Note that the in-plane equilibrium equations 8.2 decouple from the problem, since $\sigma_{\alpha\beta}$ only enters at second-order in Eq. 8.1.
3. Plane-polar coordinates would be most appropriate. The Laplace operator in plane-polar coordinates is

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}. \tag{8.7}$$

4. The general solution of the axisymmetric equation for w can be written

$$w(r) = \frac{pr^4}{64D} + C_1 r^2 \ln r + C_2 r^2 + C_3 \ln r + C_4, \tag{8.8}$$

   The boundary conditions at $r = a$ provide two equations for the unknown constants C_1, C_2, C_3 and C_4; for the other two equations, consider the shear force Q and the behaviour of w as $r \to 0$. The final solution is

$$w = \frac{p}{64D}\left(a^2 - r^2\right)^2. \tag{8.9}$$

5. The solution for the displacement is

$$w = \frac{p}{64D}\left(a^2 - r^2\right)\left(\frac{5+\nu}{1+\nu}a^2 - r^2\right). \tag{8.10}$$

6. Recall that the stress tensor $\sigma_{\alpha\beta}$ can be written in terms of the in-plane stress tensor $\sigma^o_{\alpha\beta}$, moment tensor $m_{\alpha\beta}$ and through-thickness coordinate z:

$$\sigma_{\alpha\beta} = \sigma^o_{\alpha\beta} + z m_{\alpha\beta}. \tag{8.11}$$

Problem 2

1. Refer to Eqs. 8.1–8.2 and neglect the bending term. From lectures (section 6.2, page 92), the system is closed using the constitutive relations:

$$\sigma^o_{\alpha\beta} = \frac{E}{1-\nu^2}\left[(1-\nu)E_{\alpha\beta} + \nu\delta_{\alpha\beta}E_{\gamma\gamma}\right], \tag{8.12}$$

where $E_{\alpha\beta}$ is the in-plane strain tensor. We can write $E_{\alpha\beta}$ in terms of the displacements using the kinematic relations:

$$E_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u^o_\alpha}{\partial x_\beta} + \frac{\partial u^o_\beta}{\partial x_\alpha} + \frac{\partial w}{\partial x_\alpha}\frac{\partial w}{\partial x_\beta}\right). \tag{8.13}$$

2. The stress components in the polar coordinate system are σ^o_{rr}, $\sigma^o_{r\theta}$ and $\sigma^o_{\theta\theta}$; axisymmetry then implies that $\sigma^o_{r\theta} = 0$ and σ^o_{rr}, $\sigma^o_{\theta\theta}$ are functions of r only. Similarly, the only non-zero strain components are $E_{rr}(r)$ and $E_{\theta\theta}(r)$, and the in-plane displacement is purely radial: $u_r = u_r(r)$ and $u_\theta = 0$. Using the formulae for divergence and gradient in polar coordinates, Eqs. 8.1–8.2 become

$$-h\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\sigma^o_{rr}\frac{\mathrm{d}w}{\mathrm{d}r}\right) = p, \tag{8.14}$$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\sigma^o_{rr}\right) - \frac{\sigma^o_{\theta\theta}}{r} = 0. \tag{8.15}$$

The constitute relations remain unchanged with the indices $\{\alpha, \beta\} \in \{r, \theta\}$; the kinematic relations reduce to

$$E_{rr} = \frac{\mathrm{d}u_r}{\mathrm{d}r} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2, \quad E_{\theta\theta} = \frac{u_r}{r}. \tag{8.16}$$

3. Show that the out-of-plane equilibrium equation simplifies to

$$-\frac{Eh}{2(1-\nu^2)}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^3\right] = p. \tag{8.17}$$

The relevant solution is

$$w = \frac{3}{4}\left[\frac{p(1-\nu^2)}{Eh}\right]^{1/3}\left(a^{4/3} - r^{4/3}\right). \tag{8.18}$$

4. It is helpful to first eliminate the radial displacement u_r in favour of the stress components σ^o_{rr} and $\sigma^o_{\theta\theta}$ using the constitutive/kinematic relations. The hoop stress $\sigma^o_{\theta\theta}$ can be written in terms of σ^o_{rr} using the in-plane equilibrium equation, leading to two equations for σ^o_{rr} and w.