Solutions of Exercises of Chapter 7

7. Solution:

Assume the original system is,

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u, \\
y &= \mathbf{C}\mathbf{x} + Du, \\
G(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D.
\end{aligned}$$

Assume a change of state from $\mathbf{x}$ to $\mathbf{z}$ using the nonsingular transformation $\mathbf{T}$,

$$\mathbf{x} = \mathbf{T}\mathbf{z}.$$

The new system matrices are,

$$\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T},\ \bar{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B},\ \bar{\mathbf{C}} = \mathbf{C}\mathbf{T},\ \bar{D} = D.$$

The transfer function is,

$$\begin{aligned}
G_z(s) &= \bar{\mathbf{C}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}} + \bar{D} \\
&= \mathbf{C}\mathbf{T}(s\mathbf{I} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T})^{-1}\mathbf{T}^{-1}\mathbf{B} + D.
\end{aligned}$$

If we factor $\mathbf{T}$ on the left and $\mathbf{T}^{-1}$ on the right of the $(s\mathbf{I} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T})^{-1}$ term, we obtain,

$$\begin{aligned}
G_z(s) &= \mathbf{C}\mathbf{T}(s\mathbf{T}\mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T})^{-1}\mathbf{T}^{-1}\mathbf{B} + D \\
&= \mathbf{C}\mathbf{T}\mathbf{T}^{-1}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{T}\mathbf{T}^{-1}\mathbf{B} + D = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + J = G(s).
\end{aligned}$$

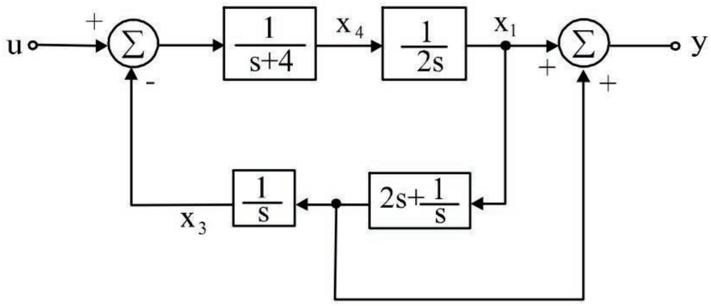
15. Solution

We are given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$. Steady-state means that $\dot{\mathbf{x}} = 0$ and a step input (or unit step) means $u = 1(t)$. Thus, assuming that the system is stable and $\mathbf{A}$ is invertible (which you can check), we have,

$$\mathbf{0} = \mathbf{A}\mathbf{x}_{ss} + \mathbf{B} \Longrightarrow \mathbf{x}_{ss} = -\mathbf{A}^{-1}\mathbf{B} = -\begin{bmatrix} -5 & 1 \\ -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/7 \\ 5/7 \end{bmatrix}.$$

16. Solution

(a) The block diagram can be simplified by moving the pick up point at x_4 to x_1. This way, H_1 will be changed to $2s + 1/s$ and we obtain the following block diagram:

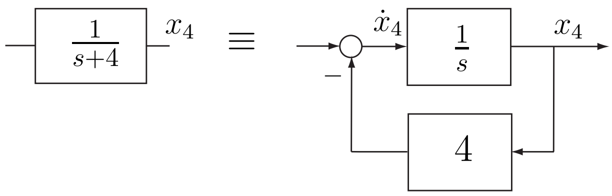


Block diagram for solution of Problem 7.16 (a).

In the next step the pick up point in the feedback loop (before $1/s$ block) will be moved to x_1. This will create a new block $1 + 2s + 1/s$ between x_1 and y and eliminate the summation. The final transfer function will be:

$$\frac{Y(s)}{U(s)} = \frac{\frac{1}{s+4}\frac{1}{2s}}{1 + \frac{1}{s+4}\frac{1}{2s}\frac{2s^2+1}{s^2}} \frac{2s^2+s+1}{s} = \frac{s(2s^2+s+1)}{2s^4+8s^3+2s^2+1}$$

(b) The block diagram includes essentially the integrators. The first order model G_1 can be written in an equivalent form to emphasise the integrator as follows:



The state and output equations can be written from the block diagram of Fig. 7.85 as follows :

$$\dot{x}_1 = 0.5x_4 \quad ; \quad \dot{x}_2 = x_1 \quad ; \quad \dot{x}_3 = x_2 + x_4 \quad ; \quad \dot{x}_4 = u - x_3 - 4x_4$$

and $y = x_1 + x_2 + x_4$. This leads to the following state-space model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix} \mathbf{x}.$$

The results can be checked for consistency using Matlab's command `ss2tf`.

22. Solution:

The natural frequency for a second-order system is related to the peak-time by the following relation (Chapter 3, Slide 41):

$$\omega_n = \frac{\pi}{t_p\sqrt{1-\zeta^2}} = \frac{1}{\sqrt{1-(0.707)^2}} = 1.414 \quad \text{rad/s}$$

Using full state feedback, we would like the a characteristic equation to be,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 2 = 0.$$

Using state feedback $u = -\mathbf{Kx}$, we get,

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -6 - k_1 & -5 - k_2 \end{bmatrix} \mathbf{x}.$$

Hence the closed-loop characteristic equation is,

$$s^2 + (5 + k_2)s + (6 + k_1) = 0.$$

Comparing coefficients, $k_1 = -4$ and $k_2 = -3$. The MATLAB command place can also be used.

25. Solution:

(a) Let's write this system in the control canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u,$$
$$y = [1 \quad 0]\mathbf{x}$$

(b) If $u = -[K_1 \quad K_2]\mathbf{x}$, the poles of the closed-loop system satisfy $\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = 0$. Thus,

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = \begin{bmatrix} s + K_1 & 4 + K_2 \\ -1 & s \end{bmatrix} = s^2 + K_1 s + 4 + K_2$$

The closed-loop characteristic equation is:

$$(s + 2 - 2j)(s + 2 + 2j) = s^2 + 4s + 8$$

Comparing coefficients, we have $K_1 = 4$ and $K_2 = 4$.

34. Solution:

(a)

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

is nonsingular. Therefore, $(\mathbf{A},\mathbf{C})$ is observable.

(b) Let,

$$\mathcal{O}_{unobs} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}(\mathbf{A}-\mathbf{BK}) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -K_1 & 1-K_2 \end{bmatrix}.$$

So if $\det(\mathcal{O}_{unobs}) = 1 - K_2 + 2K_1 = 0$, then $(\mathbf{A}-\mathbf{BK},\ \mathbf{C})$ is unobservable.

(c) $K_1 = 1 \Longrightarrow 1 - K_2 + 2 = 0 \Longrightarrow K_2 = 3$.

(d)

$$\begin{aligned} G_{ol}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \frac{s+2}{s^2+2s-1} = \frac{s+2}{(s-0.414)(s+2.414)}. \\ G_{cl}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A}+\mathbf{BK})^{-1}\mathbf{B} = \frac{s+2}{s^2+3s+2} = \frac{s+2}{(s+2)(s+1)} = \frac{1}{(s+1)}. \end{aligned}$$

The computations can be carried out using MATLAB's ss2tf command. So the unobservability is due to a **cancellation** of one of the closed-loop poles with the zero of the system. In other words, this closed-loop mode is unobservable from the output.

37. Solution:

(a) Apply Kirchhoff's voltage and current laws, with $x_1 = i_L$ and $x_2 = v_c$, we obtain,

$$\begin{aligned} L\dot{x}_1 + Rx_1 &= x_2 + RC\dot{x}_2, \\ C\dot{x}_2 &= u - x_1, \\ y &= (u - x_1)R \end{aligned}$$

Thus,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2R/L & 1/L \\ -1/C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} R/L \\ 1/C \end{bmatrix} u, \\ y &= \begin{bmatrix} -R & 0 \end{bmatrix} \mathbf{x} + Ru. \end{aligned}$$

(b) The condition for the system to be uncontrollable is $\det(\mathcal{C}) = 0$.

$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} R/L & -2R^2/L^2 + 1/LC \\ 1/C & -R/LC \end{bmatrix}. \\ \det(\mathcal{C}) &= R^2/L^2C - 1/LC^2. \end{aligned}$$

Thus, the system is controllable if $R^2 \neq L/C$.

(c) The condition for the system to be unobservable is,

$$\begin{aligned} \mathcal{O} &= \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} -R & 0 \\ 2R^2/L & -R/L \end{bmatrix}. \\ \det(\mathcal{O}) &= R^2/L. \end{aligned}$$

Since $\det(\mathcal{O}) \neq 0$ for any R, L, C except $R = 0$ or $L = \infty$, the system is observable.

46. Solution:

(a) Defining $x_1 = \theta$ and $x_2 = \dot{\theta}$, and anticipating that the measured variable in part (b) is $\dot{\theta}$, we have,

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\
y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}.
\end{aligned}$$

(b) From,

$$\begin{aligned}
\det(s\mathbf{I} - \mathbf{A} + \mathbf{LC}) &= 0, \\
\det \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \right\} &= s^2 + l_2 s + \omega^2(-l_1 + 1) = 0.
\end{aligned}$$

Using $\omega = 5$ and the specified roots for the estimator, we calculate $l_1 = -7$, and $l_2 = 20$. This result can be verified using MATLAB's `place` command.

(c) To find the transfer function from the measured value of $\dot{\theta}$, y, to the estimated value of θ, $\hat{\theta}$, we use the estimator equations,

$$\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \mathbf{C}\hat{x}) \\
&= (\mathbf{A} - \mathbf{LC})\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}y.
\end{aligned}$$

Since this is in state space form, we can now directly compute the transfer function from y to $\hat{\theta}$. It is simply,

$$\begin{aligned}
\frac{\hat{\Theta}(s)}{Y(s)} &= \begin{bmatrix} 1 & 0 \end{bmatrix} (s\mathbf{I} - \mathbf{A} + \mathbf{LC})^{-1}\mathbf{L} \\
&= \frac{-7(s - 20/7)}{s^2 + 20s + 200}.
\end{aligned}$$

(d) For controller gain $\mathbf{K} = [k_1 \; k_2]$, we require,

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = 0 \Longrightarrow s^2 + k_2 s + \omega^2 + k_1 = 0.$$

Comparing this with the specified roots equation:

$$(s + 4 + j4)(s + 4 - j4) = s^2 + 8s + 32 = 0,$$

we obtain $k_1 = 7$, and $k_2 = 8$. This result can be verified using MATLAB's `place` command.

48. Solution:

(a) From the transfer function, we can read off the elements that will give observer canonical form,

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}_o\mathbf{x} + \mathbf{B}_o u, \\
y &= \mathbf{C}_o\mathbf{x}, \\
\mathbf{A}_o &= \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}, \ \mathbf{B}_o = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \ \mathbf{C}_o = \begin{bmatrix} 1 & 0 \end{bmatrix}.
\end{aligned}
$$

(b) With $u = -[k_1\ k_2][x_1\ x_2]^T$, we want to achieve the following closed-loop characteristic equation:

$$\alpha_c(s) = (s + 2 + 2j)(s + 2 - 2j) = s^2 + 4s + 8 = 0.$$

From $\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = 0$, we obtain,

$$s^2 + 4k_2 s + 4k_1 - 4 = 0.$$

Comparing the coefficients yields $k_1 = 3$, and $k_2 = 1$. This result can be verified using MATLAB's place command.

(c) The estimator roots are determined by the equation $\alpha_e(s) = 0$. We want to find l_1 and l_2 such that,

$$\alpha_e(s) = (s + 10 + 10j)(s + 10 - 10j) = s^2 + 20s + 200.$$

$$
\begin{aligned}
\alpha_e(s) &= \det(s\mathbf{I} - \mathbf{A} + \mathbf{LC}) \\
&= \det\left(\begin{bmatrix} s & -1 \\ -4 & s \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \\
&= \det \begin{bmatrix} s + l_1 & -1 \\ -4 + l_2 & s \end{bmatrix} = s^2 + l_1 s + l_2 - 4.
\end{aligned}
$$

Comparing the coefficients yields $l_1 = 20$, $l_2 = 204$. This result can be verified using MATLAB's place command.

(d) The transfer function of the resulting compensator is,

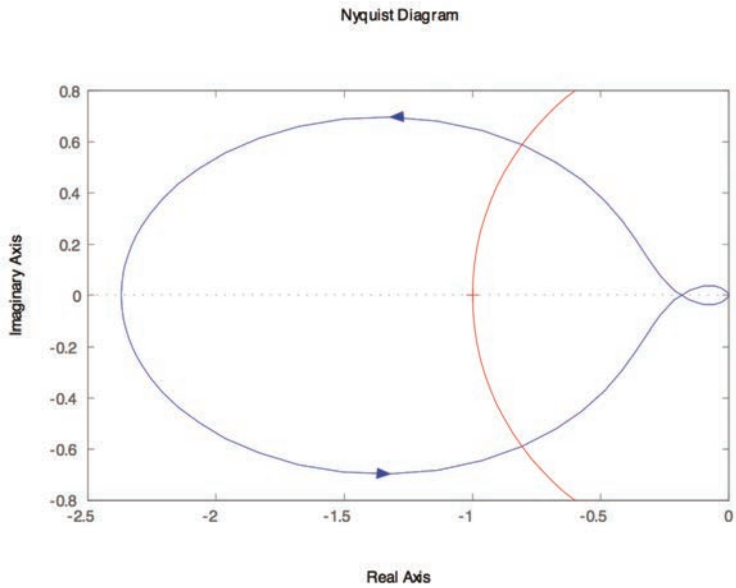
$$
\begin{aligned}
D_c(s) &= \frac{U(s)}{Y(s)} = -\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{BK} + \mathbf{LC})^{-1}\mathbf{L}, \\
&= -\begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} s + 20 & -1 \\ 212 & s + 4 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 204 \end{bmatrix} = \frac{-264s - 692}{s^2 + 24s + 292}.
\end{aligned}
$$

This result can be verified using MATLAB's ss2tf command.

(e) The loop gain is given by:

$$-D_c(s)G(s) = \frac{264s + 692}{s^2 + 24s + 292} \frac{4}{s^2 - 4}$$

Note that if we consider a positive feedback and put the negative sign in the controller $D_c(s)$ then the closed-loop poles are the zeros of $1 - D_c(s)G(s)$ and the Nyquist plot should be drawn for $-D_c(s)G(s)$. We can see the Nyquist plot for this system below from which we notice that the system has both a positive and negative gain margin (in dB). The gain can be increased by 5.46 times (GM=1/0.183=5.46), or decreased by 0.41 times (GM=1/2.37=0.41) before the system becomes unstable. From the plot, we can also see that the phase margin is about 55°.



Nyquist plot for Problem 7.48.

51. Solution

(a) The ODE of the system is $\ddot{y} + \dot{y} = 10u$. Define the state variables $x_1 = y$ and $x_2 = \dot{x}_1 = \dot{y}$, we can give the state equations as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}$$

(b) The desired characteristic equation is:

$$\alpha_c(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 3s + 9$$

The closed-loop characteristic equation is:

$$\det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}) = 10K_1 + s + 10K_2 s + s^2$$

Equating coefficients and solving gives $K_1 = 0.9$ and $K_2 = 0.2$.

(c) The desired characteristic equation is:

$$\alpha_e(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 15s + 225$$

The closed-loop characteristic equation is:

$$\det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{LC}) = (s + l_1)(s + 1) + l_2 = s^2 + (l_1 + 1)s + (l_1 + l_2)$$

Equating coefficients and solving gives $l_1 = 14$ and $l_2 = 211$.

(d) The transfer function of the controller is:

$$D_c(s) = -\boldsymbol{K}(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{BK} + \boldsymbol{LC})^{-1}\boldsymbol{L} = \frac{-(54.8s + 202.5)}{s^2 + 17s + 262}$$

58. Solution:

(a) Using feedback of the form, $u = -\mathbf{K}\mathbf{x} + \bar{N}r$, we have,

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = (s + 2 + k_1)(s + 3 + k_2) + k_1(1 - k_2) = s^2 + 6s + 18,$$

when $\mathbf{K} = [5 \ \ -4]$. This result can be verified using the MATLAB `place` command.

(b) We can find the desired value for $\bar{N}$ by setting the DC gain from r to y equal to unity. The closed-loop system equations are,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{B}(-\mathbf{Kx} + \bar{N}r) = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{B}\bar{N}r, \\ y &= \mathbf{Cx}. \end{aligned}$$

Therefore, the transfer function is,

$$T(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{BK})^{-1}\mathbf{B}\bar{N},$$

and the DC gain is simply,

$$T(0) = \mathbf{C}(-\mathbf{A} + \mathbf{BK})^{-1}\mathbf{B}\bar{N} = \frac{5}{9}\bar{N} = 1.$$

Hence, we choose $\bar{N} = \frac{9}{5}$.

(c) Change $\mathbf{A}$ to $(\mathbf{A} + \delta\mathbf{A})$, and let the value of $\bar{N}$ that keeps the tracking error at zero be N'. Then letting $T'(s)$ be the transfer function associated with the perturbed system,

$$\begin{aligned} N'^{-1} &= T'(0) = -\mathbf{C}(\mathbf{A} + \delta\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}, \\ &= -\mathbf{C}[(\mathbf{A} - \mathbf{BK})(\mathbf{I} - (\mathbf{A} - \mathbf{BK})^{-1}\delta\mathbf{A})]^{-1}\mathbf{B}, \\ &= -\mathbf{C}(\mathbf{I} - (\mathbf{A} - \mathbf{BK})^{-1}\delta\mathbf{A})^{-1}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}. \end{aligned}$$

For $\delta\mathbf{A}$ small,

$$(\mathbf{I} - (\mathbf{A} - \mathbf{BK})^{-1}\delta\mathbf{A})^{-1} = \mathbf{I} + (\mathbf{A} - \mathbf{BK})^{-1}\delta\mathbf{A}.$$

Hence,

$$N'^{-1} = \underbrace{-\mathbf{C}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}}_{\bar{N}^{-1}} - \mathbf{C}(\mathbf{A} - \mathbf{BK})^{-1}\delta\mathbf{A}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}.$$

And for arbitrary $\delta\mathbf{A}$ we arrive at,

$$N'^{-1} \neq \bar{N}^{-1}.$$

Therefore, small changes in the plant matrix $\mathbf{A}$ prevent the steady-state error from reaching zero. The control system is not robust with respect to changes in $\mathbf{A}$.

(d) Augmenting the system equations with an integrator state, x_I, the state equation become,

$$\begin{aligned}
\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_I \end{bmatrix} &= \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_I \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r, \\
y &= [\mathbf{C} \;\; \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ x_I \end{bmatrix}.
\end{aligned}$$

or with $\mathbf{z} = [\mathbf{x} \; x_I]^T$,

$$\begin{aligned}
\dot{\mathbf{z}} &= \mathbf{A}_a \mathbf{z} + \mathbf{B}_a u + \mathbf{B}_r r, \\
y &= \mathbf{C}_a \mathbf{z}.
\end{aligned}$$

Using feedback of the form $u = -\mathbf{K}\mathbf{x} - k_I x_I = -\mathbf{K}_a \mathbf{z}$, we have,

$$\det(s\mathbf{I} - \mathbf{A}_a + \mathbf{B}_a \mathbf{K}_a) = 0 \quad \text{for } s = -3, -2 \pm j\sqrt{3},$$

when $\mathbf{K}_a = \begin{bmatrix} 0.3 & 1.7 & -2.1 \end{bmatrix}$. This result can be verified using the MATLAB place command.

(e) We can show that the closed-loop DC gain from r to y is independent of $\mathbf{A}$,

$$\begin{aligned}
y_\infty &= T(0) r_\infty = [\mathbf{C} \;\; \mathbf{0}] \begin{bmatrix} -\mathbf{A} + \mathbf{B}\mathbf{K} & \mathbf{B}k_I \\ \mathbf{C} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r_\infty \\
&= [\mathbf{C} \;\; \mathbf{0}] \begin{bmatrix} * & (\mathbf{A} - \mathbf{B}\mathbf{K})^{-1} \mathbf{B}k_I [\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1} \mathbf{B}k_I]^{-1} \\ * & * \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r_\infty \\
&= [\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1} \mathbf{B}k_I][\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1} \mathbf{B}k_1]^{-1} r_\infty = r_\infty \text{ independent of } \mathbf{A}, \mathbf{B}, \mathbf{C}.
\end{aligned}$$

Remark: In the second line of the above equations, the following matrix inversion lemma is used :

$$\begin{aligned}
\begin{bmatrix} A & U \\ V & C \end{bmatrix}^{-1} &= \begin{bmatrix} I & A^{-1}U \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} A & 0 \\ 0 & C - VA^{-1}U \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ VA^{-1} & I \end{bmatrix}^{-1} \\
&= \begin{bmatrix} I & -A^{-1}U \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & (C - VA^{-1}U)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -VA^{-1} & I \end{bmatrix} \\
&= \begin{bmatrix} A^{-1} + A^{-1}U(C - VA^{-1}U)^{-1}VA^{-1} & -A^{-1}U(C - VA^{-1}U)^{-1} \\ -(C - VA^{-1}U)^{-1}VA^{-1} & (C - VA^{-1}U)^{-1} \end{bmatrix}
\end{aligned}$$

60. Solution

1. The state-space model is given by:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} 1 & \alpha \end{bmatrix}, \qquad D = 0$$

   The observability matrix is:

$$\mathcal{O} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ \alpha & 0 \end{bmatrix} \quad \Rightarrow \quad \det(\mathcal{O}) = -\alpha^2 \neq 0, \quad \text{iff } \alpha \neq 0$$

2. We should find $\boldsymbol{P} = \boldsymbol{P}^T > 0$ from the following equation:

$$\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{PA} - \boldsymbol{PBR^{-1}B^TP} + \boldsymbol{Q} = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} P_{12} & P_{22} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} P_{12} & 0 \\ P_{22} & 0 \end{bmatrix} - \begin{bmatrix} P_{11} & 0 \\ P_{12} & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2P_{12} & P_{22} \\ P_{22} & 0 \end{bmatrix} - \begin{bmatrix} P_{11}^2 & P_{11}P_{12} \\ P_{12}P_{11} & P_{12}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{aligned} &-P_{12}^2 + 1 = 0 && \Rightarrow && P_{12} = 1 \\ &2P_{12} - P_{11}^2 + 1 = 0 && \Rightarrow && P_{11} = \sqrt{3} \\ &P_{22} - P_{11}P_{12} = 0 && \Rightarrow && P_{22} = \sqrt{3} \end{aligned}$$

   Then $\boldsymbol{K} = R^{-1}\boldsymbol{B}^T\boldsymbol{P} = \begin{bmatrix} \sqrt{3} & 1 \end{bmatrix}$.

3. The characteristic polynomial is: $\alpha_o(s) = (s+3)^2 = s^2 + 6s + 9$. We have:

$$\det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{LC}) = \alpha_o(s)$$

   Therefore:

$$\det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} l_1 & l_1 \\ l_2 & l_2 \end{bmatrix}\right) = s^2 + 6s + 9$$

$$\det\left(\begin{bmatrix} s + l_1 & l_1 \\ l_2 - 1 & s + l_2 \end{bmatrix}\right) = (s + l_1)(s + l_2) + l_1(1 - l_2) = s^2 + 6s + 9$$

   which leads to $\boldsymbol{L}^T = [9 \quad -3]$.

61. Solution

a) The controllable canonical representation is given by:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 10 \end{bmatrix}$$

The augmented system with integrator is defined as:

$$\bar{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & -10 & 0 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We should solve $\det(sI - \bar{A} + \bar{B}K) = (s+2)^3$ to find the state feedback controller $K = [k_1 \quad k_2 \quad k_3]$.

$$\det\left(\begin{bmatrix} s & 0 & 0 \\ -1 & s & 0 \\ 1 & 10 & s \end{bmatrix} + \begin{bmatrix} k_1 & k_2 & k_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \det \begin{bmatrix} s+k_1 & k_2 & k_3 \\ -1 & s & 0 \\ 1 & 10 & s \end{bmatrix} = (s+k_1)s^2 + k_2 s - 10k_3 - k_3 s$$

Therefore $(s+k_1)s^2 + k_2 s - 10k_3 - k_3 s = s^3 + 6s^2 + 12s + 8$ that leads to $k_1 = 6, k_2 = 11.2$ and $k_3 = -0.8$.
If we use an observer canonical representation we will compute $k_1 = 1.12, k_2 = 0.488$ and $k_3 = -0.8$.
b) The closed-loop state space equations are:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) = Ax(t) - B[k_1 \quad k_2]x(t) - Bk_3 x_I(t) + Br(t) \\
\dot{x}_I(t) &= r(t) - y(t) = r(t) - Cx(t) - w(t) \\
u(t) &= -[k_1 \quad k_2]x(t) - k_3 x_I(t) + r(t) \\
y(t) &= Cx(t) + w(t)
\end{aligned}$$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_I(t) \end{bmatrix} = \begin{bmatrix} A - B[k_1 \quad k_2] & -Bk_3 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_I(t) \end{bmatrix} + \begin{bmatrix} B \\ 1 \end{bmatrix} r(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} w(t)$$

$$A_{cl} = \bar{A} - \bar{B}K = \begin{bmatrix} -k_1 & -k_2 & -k_3 \\ 1 & 0 & 0 \\ -1 & -10 & 0 \end{bmatrix} \quad , \quad B_{cl} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad , \quad B'_{cl} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

- Between $w(t)$ and $y(t)$: $(A_{cl}, B'_{cl}, [C \quad 0], 1)$
- Between $r(t)$ and $u(t)$: $(A_{cl}, B_{cl}, -K, 1)$