Solutions of Exercises of Chapter 6

Frequency Response Methods

6.1 Solution :

From Fig. 6.1, we see that for $\omega \ll 2$ the magnitude of $G(j\omega)$ can be approximated by a line that intersects the 0dB axis at $\omega = 8$. On the other hand, $G(j\omega) \approx K/(j\omega)$ for very small ω. Therefore, for $\omega = 8$ we have :

$$20 \log \frac{K}{8} = 0 \quad \Rightarrow \quad K = 8$$

From Fig. 6.1, we see a zero at $\omega = 2$ and another one at $\omega = 4$ (because of change of slope of 20 dB/dec at $\omega = 2$ and at $\omega = 4$). The zero at $\omega = 4$ concerns the term $(1 + as)$ in the numerator of $G(s)$ that yields $a = 1/4$. We also have a pole at $\omega = 8$, and a second pole at $\omega = 24$ (because of change of slope of -20 dB/dec at $\omega = 8$ and $\omega = 24$). The pole at $\omega = 24$ corresponds to the term $(bs + 1)$ in the denominator of $G(s)$ that yields $b = 1/24$. Therefore :

$$G(s) = \frac{8(1 + 0.5s)(1 + 0.25s)}{s(s/8 + 1)(s/24 + 1)(s/36 + 1)}$$

6.2 Solution :

The following facts can be understood from the Bode diagram :
- The slope of the magnitude in high frequencies is about -40 dB/dec. This shows a relative degree (degree of denominator-degree of numerator) of two.
- The phase diagram is monotonically decreasing, so there is no zero in the transfer function (the degree of numerator is zero).
- Although the relative degree is two, the phase does not converge to -180 degree and is decreasing in high frequencies. This shows the presence of a pure time delay $e^{-\theta s}$ in the transfer function.
- The magnitude diagram has a resonance peak which shows the presence of a pair of complex conjugate poles.

Based on the above facts the following model structure can be proposed :

$$G(s) = \frac{\gamma \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} e^{-\theta s}$$

1. The steady-state gain is 20 dB, therefore :

$$20 \log \gamma = 20 \quad \Rightarrow \quad \gamma = 10$$

2. The natural frequency is slightly greater than the resonance frequency. It can be computed as the frequency in the intersection of the two asymptotes (to low and high frequencies) in the magnitude Bode diagram. Therefore, $\omega_n = 4$ rad/s is a reasonable estimate.
3. The damping factor can be computed from the magnitude of G at $\omega_n = 4$ (around 26dB) :

$$20 \log |G(j\omega_n)| = 20 \log \gamma + 20 \log \frac{1}{2\zeta} \approx 26 \quad \Rightarrow \quad \zeta \approx 0.25$$

4. The time-delay θ can be computed from the phase diagram in high frequencies as :

$$\angle G(j\omega) \approx -\pi - \theta\omega \qquad \text{for} \quad \omega \gg \omega_n$$

where $-\pi$ is the phase contribution of the two poles (in rad) at high frequencies and $-\theta\omega$ is the phase of the time delay. If we take one arbitrary point in the phase diagram, e.g $\omega = 40$ rad/s, we have :

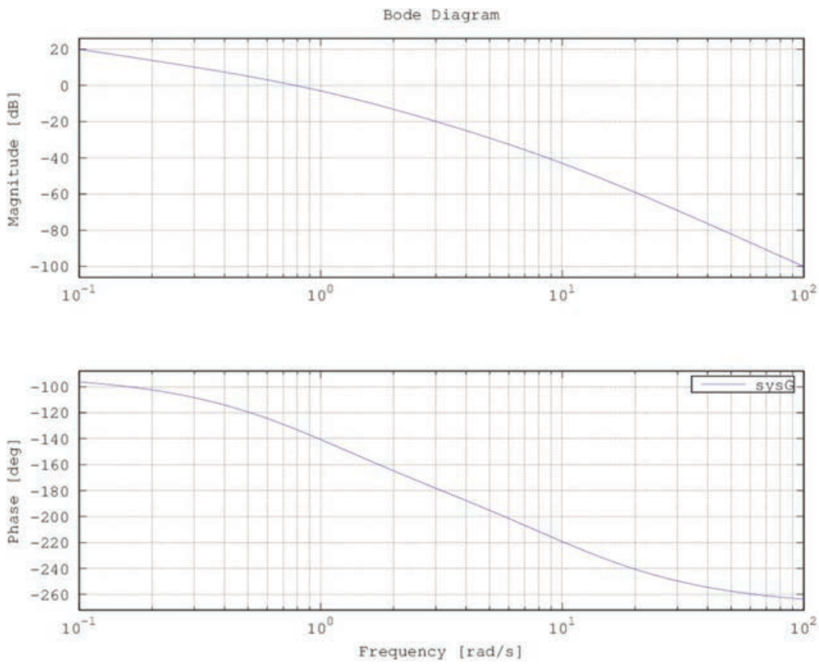
$$\angle G(j40) = -\pi - 40\theta = -405\frac{\pi}{180} \quad \Rightarrow \quad \theta = 0.098s$$

Therefore an approximate model for the system can be identified from the frequency response as :

$$G(s) = \frac{160e^{-0.1s}}{s^2 + 2s + 16}$$

6.3 Solution :

Using $k = 1$, the Bode plot of $L(s)$ is given in the following figure.



(a) The gain of $L(j\omega)$ at $\omega_{cr} = 3$ rad/s (where the phase is -180) is -20 dB. Therefore, a gain margin of 10dB can be achieved when the magnitude of L is shifted up by 10dB. This leads to $k = 10^{10/20} = 3.162$.

(b) Gain margin of 30 dB can be achieved when the magnitude of L is shifted down by 10dB, which lead to $k = 10^{-10/20} = 0.316$.

(c) The phase margin for $k = 1$ is about 45° at crossover frequency of $\omega_c \approx 0.8$ rad/s. To have a phase margin of 60°, the crossover frequency should be 0.5 rad/s for which the gain of L is about 5 dB. Therefore the magnitude of L should be shifted down for 5 dB which leads to $k = 10^{-5/20} = 0.56$

6.4 Solution

1. The pole of the system at α gives a phase variation of -90 degree (from -90 to -180 degree). At the frequency of the pole, we should have -45 degree of phase variation. As a result α is equal to the frequency at which the phase of the system is -135 degree. This leads to $\alpha = 5$. In order to compute γ, we can choose any point on the diagram and solve an equation with an unknown parameter. Let's take the magnitude of the system at 0.6 rad/s, which is 1. Therefore :

$$\left| \frac{\gamma}{j0.6 \times (j0.6 + 5)} \right| = 1 \quad \Rightarrow \quad \gamma \approx 3$$

2. Since we need an overshoot of about 30%, we choose a second-order reference model as :

$$M(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Using the specification on overshoot, we have $M_p = 0.3$ from which ζ can be computed (Chapter 3, slide 41) :

$$0.3 = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \quad \Rightarrow \quad \zeta = \sqrt{\frac{(\ln 0.3)^2}{\pi^2 + (\ln 0.3)^2}} = 0.3578$$

The natural frequency is given by (Chapter 4, slide 51) :

$$\omega_{BW} \approx \omega_n(1.85 - 1.19\zeta) = \omega_n(1.85 - 1.19 \times 0.3578) = 2 \quad \Rightarrow \quad \omega_n \approx 1.4$$

Therefore, the controller is :

$$D_c(s) = \frac{M(s)}{(1 - M(s))G(s)} = \frac{\omega_n^2 s(s + \alpha)}{(s^2 + 2\zeta\omega_n s)\gamma} = \frac{1.4^2(s + 5)}{3(s + 2 \times 0.3578 \times 1.4)} = \frac{2(s + 5)}{3(s + 1)}$$

3. The transfer function between the input disturbance and the output is :

$$\frac{Y(s)}{W(s)} = \frac{G(s)}{1 + G(s)D_c(s)} = \frac{3(s + 1)}{s(s + 5)(s + 1) + 2(s + 5)}$$

Using the final value theorem :

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s \frac{3(s + 1)}{s(s + 5)(s + 1) + 2(s + 5)} \frac{1}{s} = 0.3$$

6.5 Solution :

(a) The crossover frequency (the frequency at which the magnitude is zero dB) is $\omega_c \approx 3$ and the phase margin (180 + phase at crossover frequency) is approximately 20°. The gain margin (minus the log-magnitude at frequency where the phase is -180°) is about 7.5 dB at around 4.5 rad/s. The exact values can be computed using `margin(L)`.

(b) If the gain is doubled the magnitude diagram will shift up for $20 \log 2 = 6$ dB. The new gain margin will be 7.5-6.0=1.5 dB and the new crossover frequency will be $\omega_c \approx 4$. Then the new phase margin will be around 4°.

6.6 Solution :

First we write the equations for the crossover frequency and the phase margin :

$$|K_p G(j\omega_c)| = 1 \quad ; \quad 50\frac{\pi}{180} = \pi + \angle K_p G(j\omega_c)$$

Then K_p and ω_c can be found from the above equations. We have :

$$\frac{K_p^2}{\omega_c^2 + 100} = 1 \quad ; \quad 5\pi = 18[\pi + (-0.1\omega_c - \arctan(\omega_c/10))]$$

From the second equation $\omega_c \approx 13.4$ can be computed. This value can be obtained using the `solve` command of Matlab as follows :

```
syms x
wc=solve(5*pi-18*(pi-0.1*x-atan(x/10)))
wc=13.39440
```

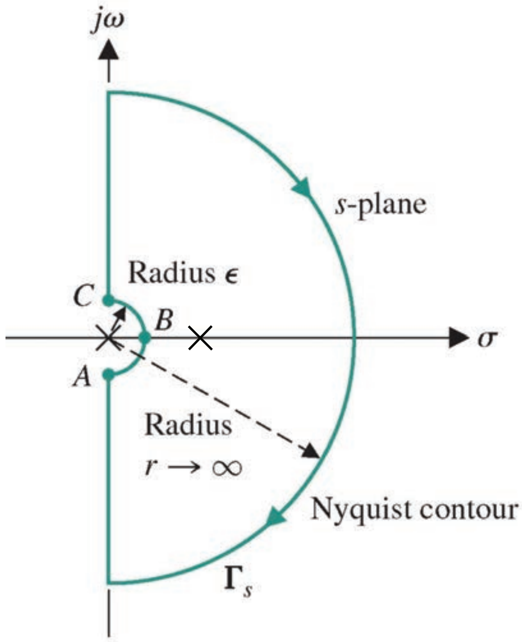
Therefore $K_p = 16.71$. The gain margin can be computed using the following equations :

$$\angle K_p G(j\omega_c) = -\pi \quad ; \quad \text{Gain margin} = |K_p G(j\omega_c)|^{-1}$$

Solving $0.1\omega_c + \arctan(\omega_c/10) = \pi$, we obtain $\omega_c = 20.2876$ and the gain margin will be $\sqrt{\omega_c^2 + 100}/K_p = 1.35$ or 2.63 dB.

6.7 Solution :

Since the plant model has a pole on the imaginary axis, the Nyquist Contour will have a small detour around zero and will contain the RHP pole of the model as it is shown in the following figure :



The sketch of the Nyquist diagram can be drawn as follows :

(a) The small semicircular detour around the pole at the origin can be represented by setting $s = \varepsilon e^{j\theta}$ and allowing θ to vary from -90° at $\omega = 0^-$ to +90° at $\omega = 0^+$. Because ε approaches zero, the mapping for $G(s)$ is

$$\lim_{\varepsilon \to 0} G(s) = \lim_{\varepsilon \to 0} \frac{1}{-\varepsilon e^{j\theta}} = \frac{-e^{-j\theta}}{\varepsilon}$$

which is a semicircle of infinite magnitude starting from -180°+90° to -180°-90° as shown in Fig. 6.7 (-180 comes from the negative sign of $G(s)$ for small s).

(b) The portion from $\omega = 0^+$ to $\omega = +\infty$ of the Nyquist contour is mapped by the function $G(j\omega)$. for very small ω, as stated before, we have :

$$G(j\omega) \approx \frac{1}{-j\omega} \quad \Rightarrow \quad |G(j\omega)| = +\infty \qquad \angle G(j\omega) \approx -270°$$

For very large value of ω we have

$$G(j\omega) \approx \frac{4j\omega}{(j\omega)^2} \quad \Rightarrow \quad |G(j\omega)| = 0 \qquad \angle G(j\omega) \approx -90°$$

The image of this portion thus intersect the negative real axis as it is shown in Fig. 6.7. The intersection point is where the imaginary part of $G(j\omega)$ is zero :

$$I_m\left[\frac{4j\omega+1}{j\omega(j\omega-1)}\right] = I_m\left[\frac{(4j\omega+1)(j\omega-\omega^2)}{\omega^4+\omega^2}\right] 0 \quad \Rightarrow \quad -4\omega^3+\omega = 0$$

which leads to $\omega_{cr} = 0.5$ and $R_e[G(j\omega_{cr})] = -4$.

(c) The semicircle portion with infinite magnitude of the Nyquist contour is mapped to zero.

(d) The portion from $\omega = -\infty$ to $\omega = 0^-$ is mapped by the function $G(-j\omega)$ and is symmetrical to the plot in item (b).

The complete Nyquist plot is shown in Fig. 6.7.

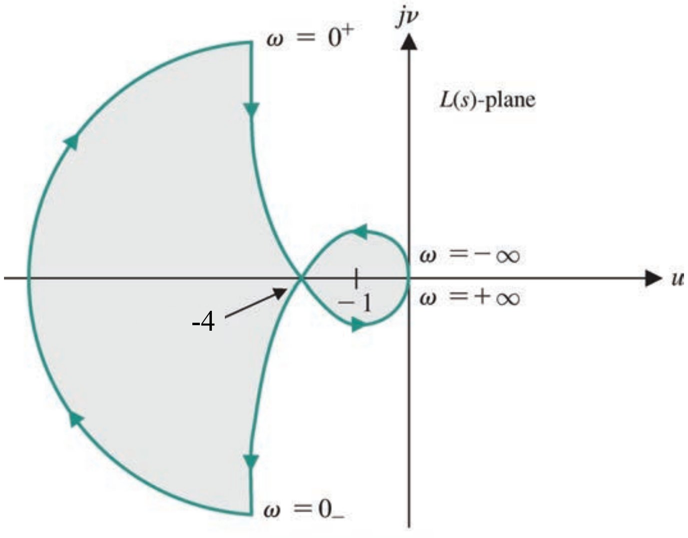


Fig. 6.7 : The Nyquist plot of $G(s)$(*Note that the scale is not respected.*)

The Nyquist plot encircles the -1 point once in a counterclockwise direction, and therefore $N = -1$. The system has one RHP pole, i.e. $P = 1$. Then the number of zeros of $1+G(s)$ in RHP is $Z = N+P = -1+1 = 0$. Thus the closed-loop system with $K_p = 1$ is stable. If $K_p = 0.25$, the Nyquist plot will pass on the critical point and closed loop system becomes unstable. Therefore the closed loop system is stable for all $K_p > 0.25$.

6.8 Solution :

In the first step we claim that the closed-loop system is stable using the complete Nyquist diagram of the open-loop system shown in the following figure :

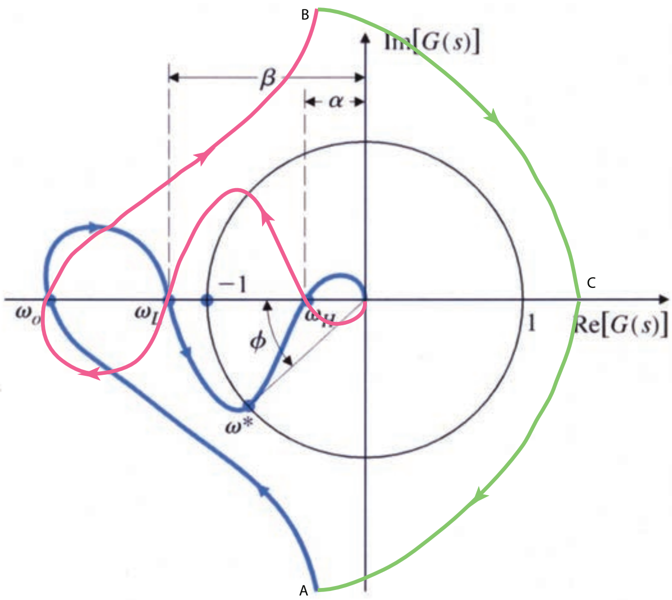


Fig. 6.8 : The complete Nyquist plot of the open-loop system

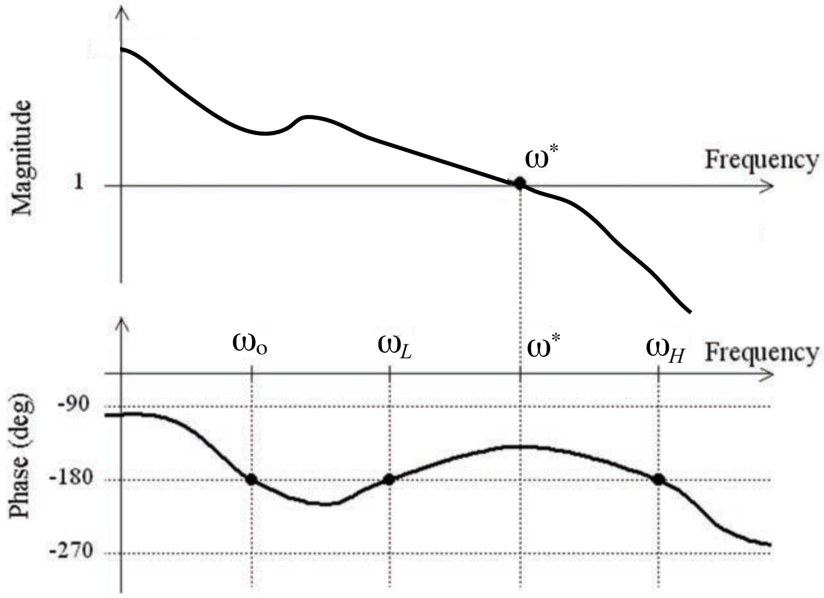
The Nyquist plot is completed using the following facts :

- The red part is the Nyquist plot for the negative frequencies which is the mirror image of the blue part with respect to the real axis.
- Since for the very low frequencies $\omega = 0^+$, the magnitude of L is very large and its phase is around -90° (point A in the figure), the open loop system has one integrator. So $L(s) \approx k/s$ for $s \to j0^+$, with $k > 0$ (for negative k the phase would be +90°).
- The image of the small detour around zero of the Nyquist contour will be a semicircle of infinite radius (green curve). Because the image of $s = \epsilon e^{j\theta}$ where $-90 < \theta < 90$ is given by $L(\epsilon e^{j\theta}) \approx \frac{k}{\epsilon} e^{-j\theta}$. It starts with a phase of +90 ° (point B) and ends up with a phase of -90° (point A). It is clear that for $\theta = 0$, L will be real and positive (point C).

Since the open-loop system is stable, we have $P = 0$. Then we compute the number of encerclement of the critical point. From Figure 6.8, we have one clockwise and one counterclockwise encirclement which leads to $N = 0$. As a result $Z = 0$ and the closed-loop system is stable.

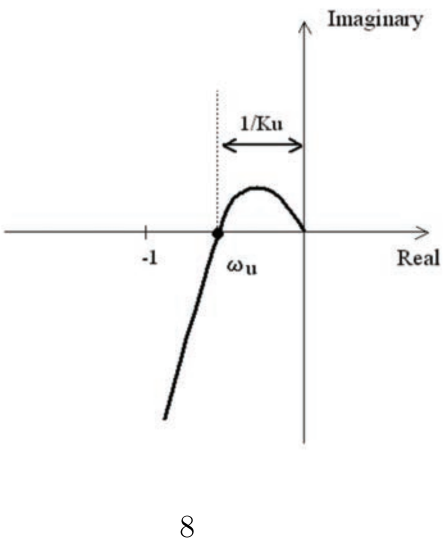
The phase margin is defined as PM $= \phi$, but there are several gain margins! If the system gain is increased (multiplied) by $1/|\alpha|$ or decreased (divided) by $|\beta|$, then the system will go unstable. This is a conditionally stable system.

For very low values of gain, the entire Nyquist plot would be shrunk, and the -1 point would occur to the left of the negative real axis crossing at ω_O, so there would be no encirclement and the system would be stable. As the gain increases, the -1 point occurs between ω_O and ω_L so there is two clockwise encirclements and the system is unstable. Further increase of the gain causes the -1 point to occur between ω_L and ω_H (as shown in the figure) so there is no encirclement and the system is stable. Even more increase in the gain would cause the -1 point to occur between ω_H and the origin where there is two clockwise encirclements and the system is unstable. The Bode plot would be vaguely like that drawn below :

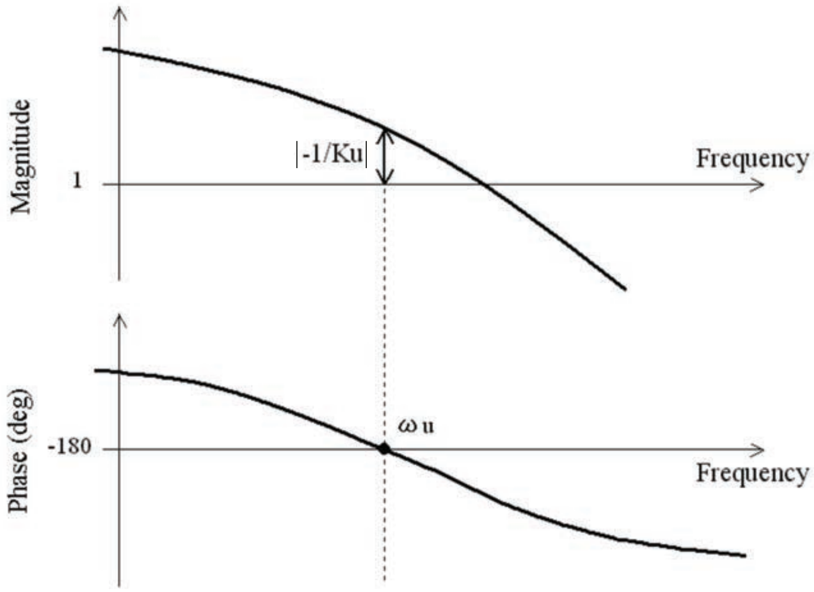


6.9 Solution :

(a) The ultimate gain brings the system to the limit of stability. So the ultimate gain causes the Nyquist diagram of $K_uG(s)$ to pass on the critical point and the system will oscillate with the ultimate period P_u . In other words, the ultimate gain K_u will be the gain margin of $G(s)$ and $\omega_u = \omega_{cr}$. Therefore : ω_u can be computed from $1 + K_uG(j\omega_u) = 0$ and $P_u = 2\pi/\omega_u$.

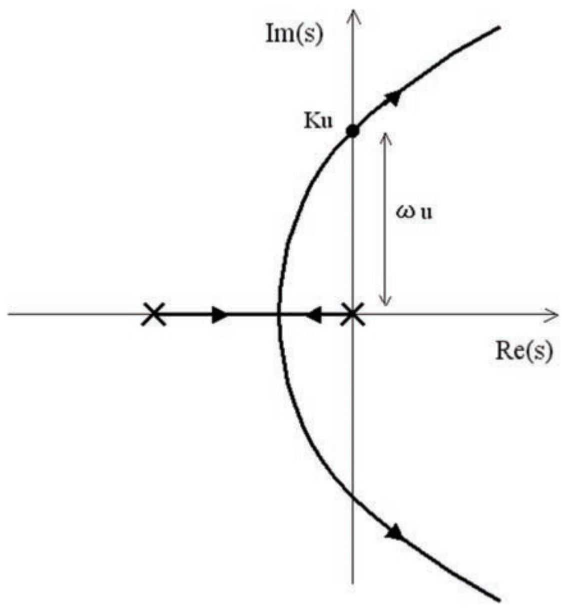


(b) At ultimate frequency the phase of $G(j\omega_u)$ is -180. As stated before, K_u will be the gain margin. Note that the magnitude diagram is not in logarithmic scale.



(c) At the limit of stability, the system will oscillate with the ultimate frequency ω_u. So it will have two complex conjugate poles on the imaginary axis in the s plan, i.e. $\pm j\omega_u$. Therefore, ω_u can be seen clearly in the root locus plan. The ultimate gain can be computed as

$$1 + K_u G(j\omega_u) = 0 \quad \Rightarrow \quad K_u = -\frac{1}{G(j\omega_u)}$$



6.10 Solution :

The step response of the process provides the parameters $b/a = 0.4$ and $1/a = 0.2$, as well as :

$$G(s) = \frac{0.4}{0.2s + 1} = \frac{2}{s + 5}$$

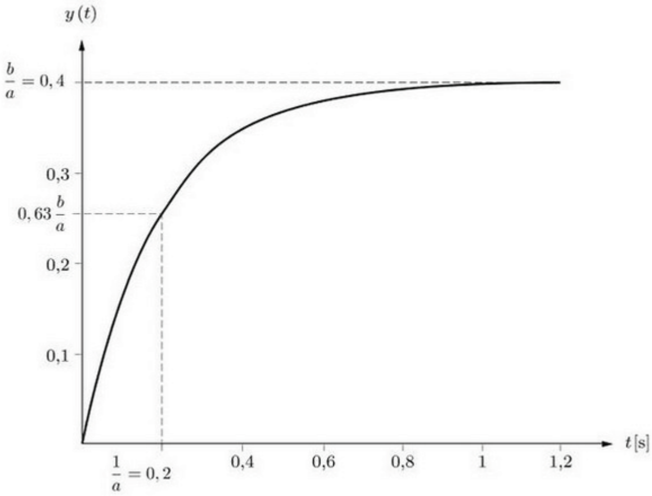


Figure 1 : Step response in closed loop.

The magnitude Bode diagram of $G(j\omega)$ is plotted in Fig. 2. The system has a pole at $1/\tau = 1/0.2 = 5$ rad/s and a steady-state gain of $20\log_{10} 0.4 = -8$ dB.

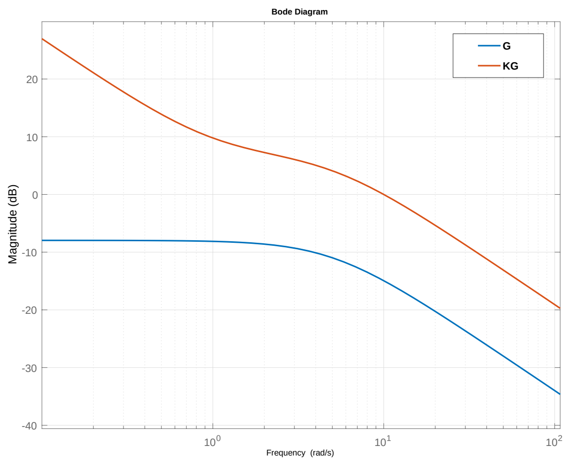


Figure 2 : Magnitude Bode diagram of the plant model G and the open-loop transfer function KG

The controller should have an integrator to meet the zero steady-state error specification. Therefore a PI controller is considered.

$$K(s) = K_p\left(1 + \frac{1}{T_i s}\right) = K_p\left(\frac{T_i s + 1}{T_i s}\right)$$

The desired crossover frequency is $\omega_c = 10$ rad/s. At this frequency, the slope of $G(j\omega)$ is -20 rad/s. The PI controller should not change this slope to ensure a phase margin of at least 60°. So the zero of the PI controller, should be designed at a frequency much smaller than ω_c to compensate the additional slope of the integrator. Therefore :

$$\frac{1}{T_i} \ll 10 \text{ rad/s} \quad \Rightarrow \quad T_i = 1$$

Now K_p should be designed such that :

$$|K(j\omega_c)G(j\omega_c)| = 1 \quad \Rightarrow \quad \left| K_p \left(\frac{(j10+1)0.4}{j10(2j+1)} \right) \right| = 1 \quad \Rightarrow \quad K_p = 5.56$$

The PI controller then is given by :

$$K(s) = K_p \left(1 + \frac{1}{T_i s} \right) = 5.56 \left(1 + \frac{1}{s} \right)$$

Alternative solution : We choose a desired open-loop transfer function as

$$L_d(s) = \frac{\omega_c}{s}$$

This transfer function guarantees a bandwidth of ω_c and a phase margin of 90° and it includes an integrator (all in one!). So we design a controller such that this desired open-loop transfer function is equal to $K(s)G(s)$:

$$K(s)G(s) = L_d(s) \quad \Rightarrow \quad K(s) = L_d(s)G^{-1}(s) = \frac{\omega_c}{s}\frac{0.2s+1}{0.4} = 5\left(1 + \frac{5}{s}\right)$$

Note that this method can be used only if $G(s)$ has no pole and zero with positive real part.

6.11 Solution :

The Bode diagram of $G(j\omega)$ is plotted in Fig. 6.7. The asymptotes are computed as follows :

- For $\omega \ll 0.86$ we have

$$G(j\omega) \approx \frac{0.043 \times 11.43 \times 2}{j\omega \times 0.86} = \frac{1.143}{j\omega}$$

  So the phase is -90° and we have a slope of -20 dB/decade in the magnitude.
- For $0.86 \ll \omega \ll 2$ we have

$$G(j\omega) \approx \frac{0.043 \times 11.43 \times 2}{-\omega^2} = \frac{-0.983}{\omega^2}$$

  So the phase is around -180° and we have a slope of -40 dB/decade.

– For $2 \ll \omega \ll 11.43$ we have

$$G(j\omega) \approx \frac{0.043 \times 11.43(-j\omega)}{-\omega^2} = \frac{0.49j}{\omega}$$

So the phase is around -270° (note that the phase cannot jump to 90°) and we have a slope of -20 dB/decade.

– For $11.43 \ll \omega$ we have

$$G(j\omega) \approx \frac{0.043 \times \omega^2}{-\omega^2} = -0.043$$

So the phase is around -180° and we have a slope of 0 dB/decade.

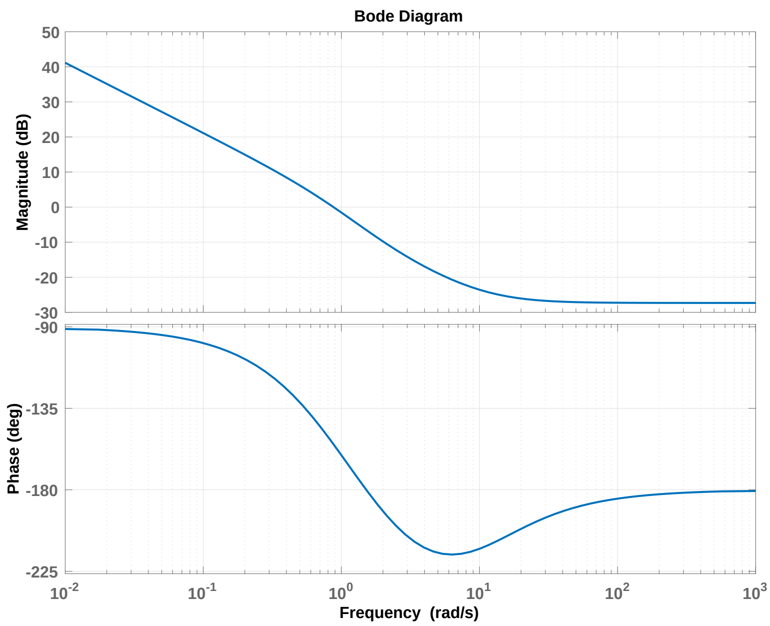


Figure 6.7 : Bode diagram of the plant model

The system is of type 1, therefore we have zero steady state error. The slope of $|G(j\omega)|$ around the crossover frequency 0.3 rad/s is about -20 dB/decade. Therefore K_p should be around -11dB or $K_p = 0.28$. The value of K_p can also be computed from the asymptote of the magnitude Bode diagram around 0.3 rad/s :

$$|K_pG(j\omega_c)| = 1 \quad \Rightarrow \quad \left|K_p\frac{1.143}{j0.3}\right| = 1 \quad \Rightarrow \quad K_p = 0.26$$

The slope of $|K_pG(j\omega)|$ around the crossover frequency is -20 dB/decade and so it has a good phase margin (greater than 60°).

6.12 Solution :

The following steps are done to design the controller for $\omega_{BW} = 30$ rad/s :

- We choose the crossover frequency $\omega_c = 30$ rad/s to guarantee a bandwidth of at least $\omega_{BW} = 30$ rad/s.
- The plant $G(s)$ has no integrator because the slope of $|G(j\omega)|$ is zero at low frequencies. Therefore, the controller should include one integrator to have zero steady-state error for tracking a step reference.
- The slope of $|G(j\omega)|$ around $\omega_c = 30$ rad/s is -40 dB/decade. By adding an integrator in the controller, the slope becomes -60 dB/decade. Therefore, the controller should have two zeros at frequencies much smaller than the crossover frequency to add 40 dB/decade to the slope of D_cG around the crossover frequency.
- A PID controller with series structure has two zeros at $-1/T_i$ and $-1/T_d$:

$$D_c(s) = K_p\left(1 + \frac{1}{T_i s}\right)(1 + T_d s) = \frac{K_p}{s}\left(s + \frac{1}{T_i}\right)(1 + T_d s)$$

  So we can choose $1/T_i \ll 30$ and $1/T_d \ll 30$. Let's take $T_i = T_d = 1$.
- The magnitude of $G(j\omega)$ at $\omega_c = 30$ rad/s is around -20 dB. Therefore, the magnitude of $D_c(j\omega)$ at ω_c should be 20 dB (or 10) as well :

$$|D_c(j\omega_c)G(j\omega_c)| = 1 \quad \Rightarrow \quad |D_c(j\omega_c)| = \left|\frac{K_p}{j\omega_c}(1 + j\omega_c)^2\right| = 10 \quad \Rightarrow \quad K_p = 0.33$$

The following steps are done to design the controller for $\omega_{BW} = 300$ rad/s :

- We choose crossover frequency $\omega_c = 300$ rad/s to guarantee a bandwidth of at least $\omega_{BW} = 300$ rad/s.
- At $\omega_c = 300$ rad/s, the slope of $|G(j\omega)|$ is -20 dB/decade. So adding an integrator the slope becomes -40 dB/decade and a PI controller (which has one zero at $-1/T_i$) is sufficient to guarantee a slope of -20 dB/decade around 300 rad/s.

$$D_c(s) = K_p\left(1 + \frac{1}{T_i s}\right)$$

- The frequency of the zero should be much less than 300 rad/s. Let's take again $T_i = 1$.
- The magnitude of $G(j\omega)$ at $\omega_c = 300$ rad/s is around -50 dB. Therefore, the magnitude of $D_c(j\omega)$ at ω_c should be 50 dB (or 316) as well :

$$|D_c(j\omega_c)G(j\omega_c)| = 1 \quad \Rightarrow \quad |D_c(j\omega_c)| = 316 \quad \Rightarrow \quad K_p = 316$$

6.13 Solution :

The magnitude Bode diagram of $G(j\omega)$ appears in Fig. 6.9.

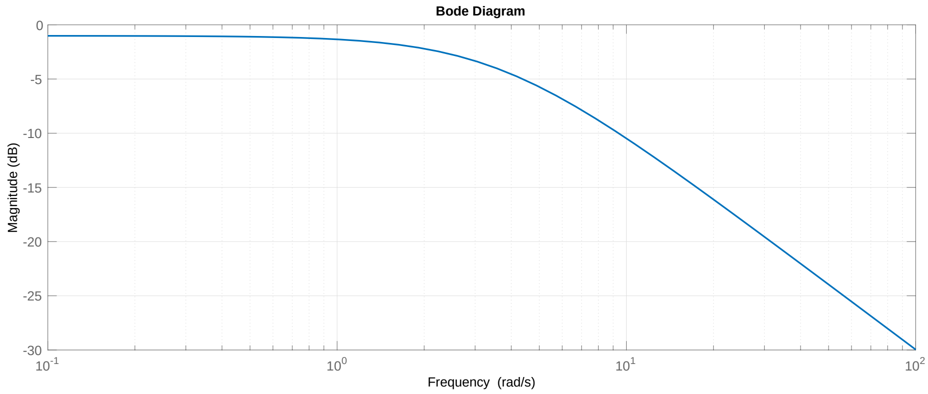


Figure 6.9 : Magnitude Bode diagram of $G(s)$

The following steps are done to design the controller :

- The controller should include one integrator to have zero steady-state error for a step disturbance.
- For a bandwidth of at least $\omega_{BW} = 12$ rad/s, we choose $\omega_c = 12$ rad/s.
- The slope of $|G(j\omega)|$ around $\omega_c = 12$ is -20 dB/decade. So adding an integrator the slope becomes -40 dB/decade and a PI controller (which has one zero at $-1/T_i$) is sufficient to guarantee a slope of -20 dB/decade around 12 rad/s.

$$D_c(s) = K_p\left(1 + \frac{1}{T_i s}\right)$$

- The frequency of the zero should be much less than 12 rad/s. Let's take $T_i = 1$.
- The magnitude of $D_c(j\omega)G(j\omega)$ at $\omega_c = 12$ rad/s should be equal to one :

$$|D_c(j\omega_c)G(j\omega_c)| = 1 \quad \Rightarrow \quad \left|K_p\left(1 + \frac{1}{j12}\right)\frac{0.89}{0.28j12 + 1}\right| = 1 \quad \Rightarrow \quad K_p = 3.92$$

Alternative solution : We choose a desired open-loop transfer function as

$$L_d(s) = \frac{\omega_c}{s}$$

This transfer function guarantees a bandwidth of at least ω_c and a phase margin of 90° and it includes an integrator (all in one !). So we design a controller such that this desired open-loop transfer function is equal to $D_c(s)G(s)$:

$$D_c(s)G(s) = L_d(s) \quad \Rightarrow \quad D_c(s) = L_d(s)G^{-1}(s) = \frac{12}{s}\left(\frac{0.28s + 1}{0.89}\right) = 3.77\left(1 + \frac{3.57}{s}\right)$$

Note that this method can be used only if $G(s)$ has no pole and zero with positive real part.

6.14 Solution

The following steps are done to design the controller :

- We choose $\omega_c = 10$ rad/s to guarantee the desired closed-loop bandwidth.
- The plant model has one integrator so there is no need to have an integrator in the controller.
- We consider the following controller structure :

$$D_c(s) = K\frac{1+\tau\alpha s}{1+\tau s}$$

- To have a steady-state error of 0.05 for a ramp reference, the steady-state gain of the open-loop transfer function (the velocity constant) should be $K_v = 1/0.05 = 20$. Therefore :

$$K_v = \lim_{s\to 0} sD_c(s)G(s) = 20K = 20 \quad \Rightarrow \quad K = 1$$

- We compute the magnitude and the phase of $KG(j\omega_c)$:

$$|KG(j\omega_c)| = \left|\frac{40}{j10(j10+2)}\right| = 0.39 = -8.13dB$$

$$\angle KG(j\omega_c) = -90^\circ - \arctan 5 = -168.7^\circ$$

- The contributions of the lead compensator for magnitude and phase are :

$$\sqrt{c} = 8.13 \text{ dB} = 2.56 \quad \text{and} \quad p = \tan(45-(180-168.7)) = 0.667$$

- Now we compute α by solving the following equation :

$$(p^2-c+1)\alpha^2+2p^2c\alpha+p^2c^2+c^2-c = 0 \quad \Rightarrow \quad -5.1\alpha^2+5.83\alpha+55.5 = 0 \quad \Rightarrow \quad \alpha = 3.92$$

- Then τ is compute as follows :

$$\tau = \frac{1}{\omega_c}\sqrt{\frac{1-c}{c-\alpha^2}} = 0.08$$

- Finally the controller is given as :

$$D_c(s) = K\frac{1+\tau\alpha s}{1+\tau s} = \frac{1+0.311s}{1+0.08s}$$

6.15 Solution

1. The slope of the magnitude Bode diagram is -40 dB/dec everywhere, so the system is a double integrator. Since the phase is decreasing monotonicly, the system includes a time delay. Therefore :

$$G(s) = \frac{Ke^{-\theta s}}{s^2}$$

The gain at $\omega = 2$ rad/s is 1, which leads to $K = 4$. At $\omega = 3.5$ the phase is -200°, therefore :

$$\angle e^{-\theta j 3.5} = -20^\circ \quad \Rightarrow \quad 3.5\theta = \frac{20\pi}{180} \quad \Rightarrow \quad \theta = 0.1$$

2. The gain contribution of the lead compensator should be 10 dB at $\omega_c = 3.5$ rad/s and its phase contribution should be $p = \tan(45 + 20) = 2.14$. Therefore :

$$\sqrt{c} = 10 \text{ dB} = 3.16 \quad \Rightarrow \quad c = 10$$

We compute α from this equation :

$$(p^2 - c + 1)\alpha^2 + 2p^2 c\alpha + p^2 c^2 + c^2 - c = 0 \quad \Rightarrow \quad -4.4\alpha^2 + 92\alpha + 550 = 0 \quad \Rightarrow \quad \alpha = 25.75$$

Then τ is computed as follows :

$$\tau = \frac{1}{\omega_c}\sqrt{\frac{1 - c}{c - \alpha^2}} = 0.0335$$

3. The controller is given by :

$$D_c(s) = \frac{1 + 0.86s}{1 + 0.0335s}$$