Solutions of Exercises of Chapter 4

4. Solution:

(a)

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{\dfrac{A}{s(s+a)}}{1+\dfrac{A}{s(s+a)}} = \frac{A}{s^2+as+A},$$

$$\frac{dT}{dA} = \frac{(s^2+as+A)-A}{(s^2+as+A)^2},$$

$$S_A^T = \frac{A}{T}\frac{dT}{dA} = \frac{A(s^2+as+A)}{A}\frac{s^2+as}{(s^2+as+A)^2} = \frac{s(s+a)}{s(s+a)+A}.$$

(b)

$$\frac{dT}{da} = \frac{-sA}{(s^2+as+A)^2}.$$

$$\frac{a}{T}\frac{dT}{da} = \frac{a(s^2+as+A)}{A}\frac{-sA}{(s^2+as+A)^2}.$$

$$S_a^T = \frac{-as}{s(s+a)+A}.$$

(c) In this case,

$$T(s) = \frac{G(s)}{1+\beta G(s)},$$

$$\frac{dT}{d\beta} = \frac{-G(s)^2}{(1+\beta G(s))^2},$$

$$\frac{\beta}{T}\frac{dT}{d\beta} = \frac{\beta(1+\beta G)}{G}\frac{-G^2}{(1+\beta G)^2} = \frac{-\beta G}{1+\beta G},$$

$$S_\beta^T = \frac{\dfrac{-\beta A}{s(s+a)}}{1+\dfrac{\beta A}{s(s+a)}} = \frac{-\beta A}{s(s+a)+\beta A}.$$

10. Solution:

(a)

$$\begin{aligned}
D_c(s)G(s) &= \frac{K(s+\alpha)^2}{(s^2+\omega_o^2)s(s+1)}, \\
\frac{E(s)}{R(s)} &= \frac{1}{1+D_cG}, \\
&= \frac{s(s+1)(s^2+\omega_o^2)}{(s^2+\omega_o^2)s(s+1)+K(s+\alpha)^2}.
\end{aligned}$$

The gain of this transfer function is zero at $s = \pm j\omega_o$ and we expect the error to be zero if R is a sinusoid at that frequency. More formally, let $R(s) = \dfrac{\omega_n}{s^2+\omega_n^2}$ then

$$E(s) = \frac{s(s+1)(s^2+\omega_o^2)}{(s^2+\omega_o^2)s(s+1)+K(s+\alpha)^2}\,\frac{\omega_n}{s^2+\omega_n^2}.$$

Assuming the (closed-loop) system is stable, then if $\omega_n \neq \omega_o$, $E(s)$ has a pole on the imaginary axis and the FVT does not apply. The final error will NOT be zero in this case. However, if $\omega_n = \omega_o$ we *can* use the FVT and

$$e_{ss} = \lim_{s\to 0} sE(s) = 0$$

(b) To test for stability, the characteristic equation is,

$$s^4 + (K+\omega_o^2)s^2 + s^3 + (\omega_o^2 + 2\alpha K)s + K\alpha^2 = 0$$

Using the Routh array

$$\begin{array}{lccc}
s^4: & 1 & \omega_o^2+K & K\alpha^2 \\
s^3: & 1 & (\omega_o^2+2\alpha K) & \\
s^2: & K(1-2\alpha) & K\alpha^2 & \\
s^1: & \omega_o^2+2\alpha K - \dfrac{\alpha^2}{(1-2\alpha)} & & \\
s^0: & K\alpha^2 & &
\end{array}$$

If $\alpha = 0.25$, we must have $K > 0$, and $K >$ -1.75

15. Solution:

(a) The transfer function between the reference signal and the tracking error is:

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)D_c(s)} = \frac{1}{1+\frac{1}{s^2}\frac{10(s+2)}{s+5}} = \frac{s^2(s+5)}{s^2(s+5)+10(s+2)}$$

For a step reference signal we have $R(s) = 1/s$ and:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = 0$$

For a ramp reference signal we have $R(s) = 1/s^2$ and:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = 0$$

For a parabolic reference signal we have $R(s) = 1/s^3$ and:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s+5}{s^2(s+5)+10(s+2)} = \frac{5}{20} = 0.25$$

(b) The transfer function between the disturbance w and the error signal is:

$$\frac{E(s)}{W(s)} = -\frac{G(s)}{1+G(s)D_c(s)} = -\frac{1/s^2}{1+\frac{1}{s^2}\frac{10(s+2)}{s+5}} = -\frac{(s+5)}{s^2(s+5)+10(s+2)}$$

Therefore, for a step disturbance $W(s) = 1/s$, we have

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} -\frac{(s+5)}{s^2(s+5)+10(s+2)} = -0.25$$

Note that, in practice when we talk usually about the absolute value of the steady-state error. Therefore, $e_{ss} = 0.25$ is fine as well.

16. Solution:

(a) Yes, because $G(s)D_c(s)$ includes an integrator. It can be confirmed by computing the steady-state error for a step reference signal $R(s) = 1/s$. The transfer function between the reference signal and the tracking error is:

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)D_c(s)} = \frac{1}{1+\frac{1}{s(s+2)}\frac{160(s+4)}{s+30}} = \frac{s(s+2)(s+30)}{s(s+2)(s+30)+160(s+4)}$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s(s+2)(s+30)}{s(s+2)(s+30)+160(s+4)} = 0$$

For a ramp reference signal we have $R(s) = 1/s^2$ and:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{(s+2)(s+30)}{s(s+2)(s+30)+160(s+4)} = \frac{60}{160\times 4} = 0.09375$$

(b) No, because the *controller* has no integrator. It can be confirmed by computing the steady-state error. The transfer function between the disturbance w and the error signal is:

$$\frac{E(s)}{W(s)} = -\frac{G(s)}{1 + G(s)D_c(s)} = -\frac{(s+30)}{s(s+2)(s+30)+160(s+4)}$$

Therefore, for a step disturbance $W(s) = 1/s$, we have

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} -\frac{(s+30)}{s(s+2)(s+30)+160(s+4)} = -0.046$$

(c) The closed loop transfer function is:

$$T(s) = \frac{G(s)D_c(s)}{1 + G(s)D_c(s)} = \frac{160(s+4)}{s(s+a)(s+30)+160(s+4)}$$

where a was inserted for the pole at -2. By definition

$$\mathcal{S}_a^T = \frac{a}{T}\frac{\partial T}{\partial a}$$

But:

$$\frac{\partial T}{\partial a} = -\frac{160(s+4)s(s+30)}{[s(s+30)(s+a)+160(s+4)]^2}$$

therefore, the sensitivity at $a = 2$ is:

$$\mathcal{S}_a^T = -\frac{a}{T}\frac{160(s+4)s(s+30)}{[s(s+30)(s+a)+160(s+4)]^2} = -\frac{2s(s+30)}{s(s+2)(s+30)+160(s+4)}$$

(d) The real tracking error is $e(t) = r(t) - y(t)$ which is not shown in the block diagram. Therefore, $E(s) = R(s) - Y(s)$ and

$$E(s) = R(s) - Y(s) = \left[1 - \frac{Y(s)}{R(s)}\right] R(s) = \left[1 - \frac{G(s)D_c(s)}{1 + H(s)G(s)D_c(s)}\right] R(s)$$

For a step reference signal, $R(s) = 1/s$ and

$$\begin{aligned}\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) &= \lim_{s\to 0} s\left[1 - \frac{G(s)D_c(s)}{1 + H(s)G(s)D_c(s)}\right]\frac{1}{s} \\ &= 1 - \lim_{s\to 0}\frac{160(s+4)(s+20)}{s(s+2)(s+30)(s+20)+160(s+4)20} = 0\end{aligned}$$

For a ramp reference signal, $R(s) = 1/s^2$ and

$$\begin{aligned}\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) &= \lim_{s\to 0}\left[1 - \frac{G(s)D_c(s)}{1 + H(s)G(s)D_c(s)}\right]\frac{1}{s} \\ &= \lim_{s\to 0}\frac{s(s+2)(s+30)(s+20)+160(s+4)20 - 160(s+4)(s+20)}{s(s+2)(s+30)(s+20)+160(s+4)20}\frac{1}{s} \\ &= \lim_{s\to 0}\frac{s(s+2)(s+30)(s+20) - 160(s+4)s}{s(s+2)(s+30)(s+20)+160(s+4)20}\frac{1}{s} = \frac{1200-640}{20(640)} = 0.04375\end{aligned}$$

For a step disturbance, we have $W(s) = 1/s$ and $E(s) = R(s) - Y(s)$ with $R(s) = 0$ and:

$$Y(s) = \frac{G(s)}{1 + H(s)G(s)D_c(s)} W(s)$$

Therefore,

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{s\to 0} sE(s) = \lim_{s\to 0} -sY(s) = \lim_{s\to 0} \frac{-G(s)}{1 + H(s)G(s)D_c(s)} \\ &= \lim_{s\to 0} \frac{-(s+20)(s+30)}{s(s+2)(s+30)(s+20) + 160(s+4)20} = -0.046\end{aligned}$$

Remark: Since the steady-state gain of $H(s)$ is one, it does not change the steady-state errors for step reference and step disturbance signals.

30. Solution:

(a) $\dfrac{Y(s)}{R(s)} = \dfrac{10(k_I + k_P s)}{s[s(s+1)+20] + 10(k_I + K_P s)}$

(b) $\dfrac{Y(s)}{W(s)} = \dfrac{10s}{s[s(s+1)+20] + 10(k_I + K_P s)}$

(c) The controller has an integrator and a unity feedback so the steady-state error for a step reference is zero. It can be verified using the final value theorem. For a step reference signal $R(s) = 1/s$:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{s(s^2+s+20)}{s(s^2+s+20) + 10(k_P s + k_I)} \frac{1}{s} = 0$$

For a ramp reference signal $R(s) = 1/s^2$:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{s(s^2+s+20)}{s(s^2+s+20) + 10(k_P s + k_I)} \frac{1}{s^2} = \frac{2}{k_I}$$

(d) The controller has an integrator and a unity feedback so the steady-state error for a step disturbance is zero. It can be verified using the final value theorem. For a step disturbance signal $W(s) = 1/s$:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{10s}{s[s(s+1)+20] + 10(k_I + K_P s)} \frac{1}{s} = 0$$

For a ramp reference signal $W(s) = 1/s^2$:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{10s}{s[s(s+1)+20] + 10(k_I + K_P s)} \frac{1}{s^2} = \frac{1}{k_I}$$

(e) The characteristic equation is $s^3 + s^2 + (10k_P + 20)s + 10k_I = 0$. The Routh's array is

$$\begin{array}{lcc} s^3: & 1 & 10k_P + 20 \\ s^2: & 1 & 10k_I \\ s^1: & 10k_P + 20 - 10k_I & \\ s^0: & 10k_I & \end{array}$$

For stability we must have $k_I > 0$ and $k_P > k_I - 2$.

34. Solution:

(a) The characteristic equation is:

$$Js^2 + H_y k_P = 0 \quad \Rightarrow \quad s^2 + k_P/J = 0$$

The Routh's array is:

$$\begin{array}{ll} s^2: & 1 \quad k_P \\ s^1: & 0 \end{array}$$

Since we have one zero in the first column, the system is unstable whatever the value of k_P is.

(b) The transfer function between $\Theta(s)$ and $\Theta_r(s)$ is:

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{H_r(k_P + k_D s)\frac{1}{Js^2}}{1 + H_y(k_P + k_D s)\frac{1}{Js^2}} = \frac{H_r(k_P + k_D s)}{Js^2 + H_y(k_P + k_D s)}$$

It is clear that for some values of k_P and k_D the closed loop system can be stable. In this case, $E(s) = \Theta_r(s) - \Theta(s)$ and

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \left[1 - \frac{\Theta(s)}{\Theta_r(s)}\right] \Theta_r(s)$$

For a step reference signal $\Theta_r(s) = 1/s$ and

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s \left[1 - \frac{\Theta(s)}{\Theta_r(s)}\right] \frac{1}{s} = 1 - \frac{H_r}{H_y} = 0$$

The transfer function between $W(s)$ and $\Theta(s)$ is:

$$\frac{\Theta(s)}{W(s)} = \frac{\frac{1}{Js^2}}{1 + H_y(k_P + k_D s)\frac{1}{Js^2}} = \frac{1}{Js^2 + H_y(k_P + k_D s)}$$

and the error is $E(s) = 0 - \Theta(s)$. For a step disturbance we have $W(s) = 1/s$:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \left[0 - \frac{\Theta(s)}{W(s)}\right] \frac{1}{s} = -\frac{1}{H_y k_P}$$

(c) The characteristic equation of the system with a PI controller is:

$$Js^3 + H_y k_P s + H_y k_I = 0$$

Since the coefficient of s^2 is zero using the Routh's test we can conclude that there is at least one unstable pole in closed-loop. Therefore, a PI controller cannot stabilize the system and the steady-state errors go to infinity.

(d) The characteristic equation of the system with a PID controller is:

$$Js^3 + H_y k_D s^2 + H_y k_P s + H_y k_I = 0$$

so the closed-loop system can be stabilized. The transfer function between $\Theta(s)$ and $\Theta_r(s)$ is:

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{H_r(k_P + k_I/s + k_D s)\frac{1}{Js^2}}{1 + H_y(k_P + k_I/s + k_D s)\frac{1}{Js^2}} = \frac{H_r(k_I + k_P s + k_D s^2)}{Js^3 + H_y(k_I + k_P s + k_D s^2)}$$

For a step reference signal $\Theta_r(s) = 1/s$ and

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\left[1 - \frac{\Theta(s)}{\Theta_r(s)}\right]\frac{1}{s} = 1 - \frac{H_r}{H_y} = 0$$

The transfer function between $W(s)$ and $\Theta(s)$ is:

$$\frac{\Theta(s)}{W(s)} = \frac{\frac{1}{Js^2}}{1 + H_y(k_P + k_I/s + k_D s)\frac{1}{Js^2}} = \frac{s}{Js^3 + H_y(k_I + k_P s + k_D s^2)}$$

and the error is $E(s) = 0 - \Theta(s)$. For a step disturbance we have $W(s) = 1/s$:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[0 - \frac{\Theta(s)}{W(s)}\right]\frac{1}{s} = 0$$

36. Solution:

(a) From step response: $L = \tau_d \simeq 0.65$ sec

$$R = \frac{1}{\tau} \simeq \frac{0.2}{1.25 - 0.65} = 0.33 \quad \text{sec}^{-1}.$$

From Table 4.1:

Controller Gain P: $K = \frac{1}{RL} = 4.62$,

PI : $K = \frac{0.9}{RL} = 4.15$ $T_I = \frac{L}{0.3} = 2.17$,

PID: $K = \frac{1.2}{RL} = 5.54$ $T_I = 2L = 1.3 T_D = 0.5L = 0.33$.

(b) From the impulse response: $P_u \simeq 2.33$ sec. and from Table 4.2:

Controller Gain P: $K = 0.5K_u = 4.28$,

PI : $K = 0.45K_u = 3.85$ $T_I = \frac{1}{1.2}P_u = 1.86$,

PID: $K = 0.6K_u = 5.13$ $T_I = \frac{1}{2}P_u = 1.12 T_D = \frac{1}{8}P_u = 0.28$.

37. Solution:

(a) From the transfer function: $L = \tau_d \simeq 2$ sec

$$R = \frac{1}{3} \simeq 0.33 \quad \text{sec}^{-1}.$$

From Table 4.1:

$$\begin{array}{llll} \text{Controller Gain} \quad P & : & & K = \frac{1}{RL}1.5, \\ PI: & K = \frac{0.9}{RL} = 1.35 & T_I = \frac{L}{0.3} = 6.66, & \\ PID: & K = \frac{1.2}{RL} = 1.8 & T_I = 2L = 4 \; T_D = 0.5L = 1.0. & \end{array}$$

(b) From the impulse response: $P_u \simeq 7$ sec From Table 4.2:

$$\begin{array}{lll} \text{Controller Gain} \quad \text{P} & : & K = 0.5K_u = 1.52, \\ \text{PI} & : & K = 0.45K_u = 1.37 \;\; T_I = \frac{1}{1.2}P_u = 5.83, \\ \text{PID} & : & K = 0.6K_u = 1.82 \;\; T_I = \frac{1}{2}P_u = 3.5T_D = \frac{1}{8}P_u = 0.875. \end{array}$$

41. Solution:

The Laplace transform of the step response of the system is computed as:

$$Y(s) = \frac{e^{-30s}}{100s+1}\frac{1}{s} = e^{-30s}\frac{0.01}{s(s+0.01)} = e^{-30s}\left(\frac{1}{s} - \frac{1}{s+0.01}\right)$$

In the time-domain it is equal to the $1 - e^{-0.01t}$ delayed by 30 sec. From the following figure, we can drive the parameters $R = 0.01$ and $L = 30$. The P, PI and PID controllers based on the ZN tuning rule are:

P controller: $\quad K(s) = \frac{1}{RL} = 3.33$

PI controller: $\quad K(s) = \frac{0.9}{RL}\left(1 + \frac{1}{3.3Ls}\right) = 3\left(1 + \frac{0.01}{s}\right)$

PID controller: $\quad K(s) = \frac{1.2}{RL}\left(1 + \frac{1}{2Ls} + 0.5Ls\right) = 4\left(1 + \frac{0.017}{s} + 15s\right)$

The integral term is welcome since the system we want to control does not have one. This term will eliminate steady-state error in the case of a constant disturbance. The derivative term is necessary in order to tackle the destabilizing effect of the delay of 30s. Our choice is, hence, the PID controller, a choice that will be validated from a detailed analysis of the stability of the system in closed loop.

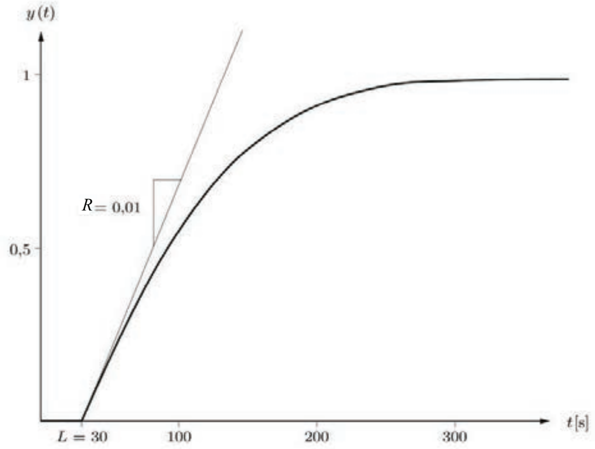

FIGURE 1 – Step response in open-loop with the tangent line of maximum slope

For a proportional controller, the transfer function between $Y_c(s)$ and $E(s)$ is:

$$\frac{E(s)}{Y_c(s)} = \frac{1}{1 + K_p \frac{e^{-30s}}{100s+1}}$$

Therefore, for a step reference input we have:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s \frac{1}{1 + K_p \frac{e^{-30s}}{100s+1}} \frac{1}{s} = \frac{1}{1 + K_p} = 0.231$$

42. Solution:

The step response together with the tangent line of maximum slope is shown in Fig. 2. From this figure we can find $L \approx 2.7$ and $R \approx 5/6 = 0.83$. Therefore, the PID controllers based on the ZN tuning rule is:

$$K(s) = \frac{1.2}{RL}\left(1 + \frac{1}{2Ls} + 0.5Ls\right) = 0.53\left(1 + \frac{0.185}{s} + 1.35s\right)$$

The parameters of the first-order model with delay $G(s)$ are $\gamma = 5, \tau = 6$ and $\theta = 2.7$ and

$$G(s) = \frac{\gamma e^{-\theta s}}{\tau s + 1} = \frac{5e^{-2.7s}}{6s + 1}$$

The settling time for the step response of a first-order system with time constant τ_m is about $4\tau_m$. A settling time of 10 seconds is equivalent of $\tau_m \approx 10/4 = 2.5$. The reference model is then chosen as:

$$M(s) = \frac{e^{-2.7s}}{2.5s + 1}$$

Then, the controller is given by:

$$K(s) = \frac{M(s)}{G(s)(1 - M(s))} = \frac{1 + 6s}{5(1 + 2.5s - e^{-2.7s})}$$

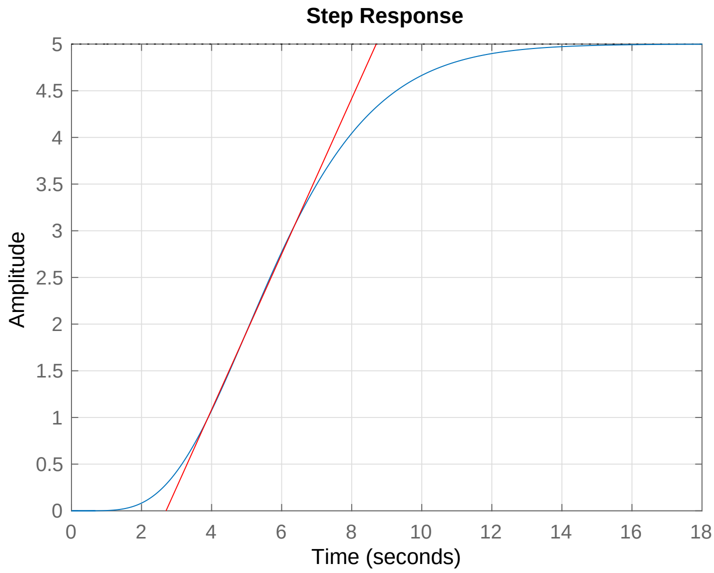

FIGURE 2 – Step response of the open-loop system with its tangent of maximum slope

If we approximate the delay with a first-order Pade approximation, we obtain:

$$\begin{aligned} K_p &= \frac{\theta/2+\tau}{\gamma(\tau_m+\theta)} = \frac{1.35+6}{5(2.5+2.7)} = 0.28 \\ T_i &= \theta/2+\tau = 1.35+6 = 7.35 \\ T_d &= \frac{\tau\theta}{\theta+2\tau} = 1.1 \end{aligned}$$

43. Solution:

From the step response, we have $K = 0.5, t_p = 0.8635$ and $y(t_p) = 0.6269$. Therefore, $\gamma = K = 0.5$ and

$$M_p = \frac{y(t_p)-K}{K} = 0.2538 = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \quad \Rightarrow \quad \zeta = \sqrt{\frac{(\ln M_p)^2}{\pi^2+(\ln M_p)^2}} = 0.4$$

$$\omega_n = \frac{\pi}{t_p\sqrt{1-\zeta^2}} \approx 4$$

The desired bandwidth is $1.2\omega_n = 4.8$ which leads to $\tau_m = 1/4.8 = 0.21$. The parameters of the PID controller for MRC are :

$$\begin{aligned} K_p &= \frac{2\zeta}{\gamma\omega_n\tau_m} = \frac{0.8}{0.5\times 4\times 0.21} = 1.9 \\ T_i &= \frac{2\zeta}{\omega_n} = 0.2 \\ T_d &= \frac{1}{2\zeta\omega_n} = 0.3125 \end{aligned}$$