Solutions of Exercises of Chapter 3

10. Solution:

There are only two cases to consider.

Case (a) For the case $t \leq 0$, there is no overlap between the two functions $u(t-\tau)$ and $h(\tau)$, so the output is zero: $y_1(t) = 0$

Case (b) For the case $t > 0$, the output of the system is given by:

$$
\begin{aligned}
y_2(t) &= \int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t 2(t-\tau)e^{-2(t-\tau)}(1)d\tau \\
&= \left[\frac{2(t-\tau)e^{-2(t-\tau)}}{2}\right]_0^t + 2\int_0^t \frac{e^{-2(t-\tau)}}{2}d\tau \\
&= \left[(t-\tau)e^{-2(t-\tau)} + \frac{e^{-2(t-\tau)}}{2}\right]_0^t = \frac{1}{2} - te^{-2t} - \frac{1}{2}e^{-2t}
\end{aligned}
$$

17. Solution:

(a) If we blindly compute the DC gain $G(0) = \frac{2}{-2} = -1$. This answer is **not** correct as explained in part (b). This is because the DC gain is not defined for an unstable system and the output of the system is unbounded.

(b) $\lim_{t\to\infty} y(t) = ?$

The poles of the system are: $s^2 + s - 2 = 0 \Longrightarrow s = 1, -2$.

Since the system has an *unstable* pole, the Final Value Theorem is *not* applicable. The output is unbounded.

26. Solution:

$$\begin{aligned}
\frac{Y(s)}{R(s)} &= \frac{K}{s^2+2s+K} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}, \\
\omega_n &= \sqrt{K}, \\
\zeta &= \frac{2}{2\omega_n} = \frac{1}{\sqrt{K}}. \qquad (1)
\end{aligned}$$

In order to have an overshoot of no more than 10%:

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \leq 0.10.$$

Solving for ζ :

$$\zeta = \sqrt{\frac{(\ln M_p)^2}{\pi^2+(\ln M_p)^2}} \geq 0.591.$$

Using (1) and the solution for ζ:

$$K = \frac{1}{\zeta^2} \leq 2.86,$$
$$\therefore 0 < K \leq 2.86.$$

27. Solution:

$$\frac{Y(s)}{R(s)} = \frac{100K}{s^2 + (25+a)s + 25a + 100K} = \frac{100K}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Using the given information:

$$\begin{aligned} R(s) &= \frac{1}{s} \qquad \text{unit step,} \\ M_p &\leq 25\%, \\ t_s &\leq 0.1\,\text{sec}. \end{aligned}$$

Solve for ζ :

$$\begin{aligned} M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}}, \\ \zeta &= \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}} \geq 0.4037. \end{aligned}$$

Solve for ω_n:

$$e^{-\zeta\omega_n t_s} = 0.01 \qquad \text{For a 1\% settling time.}$$

$$t_s \leq \frac{4.605}{\zeta\omega_n} = 0.1,$$
$$\Longrightarrow \omega_n \approx 114.07.$$

Now find a and K :

$$2\zeta\omega_n = (25 + a),$$
$$a = 2\zeta\omega_n - 25 = 92.10 - 25 = 67.10,$$
$$\omega_n^2 = (25a + 100K),$$
$$K = \frac{\omega_n^2 - 25a}{100} \approx 113.34.$$

33. Solution: The equation of motion is

$$m\ddot{x} + b\dot{x} + kx = F.$$

The transfer function is

$$\frac{X(s)}{F(s)} = G(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}.$$

In this case

$$\begin{aligned} 2G(0) &= 0.1, \\ 2\left(\frac{1}{k}\right) &= 0.1, \\ k &= 20. \end{aligned}$$

We observe that

$$\omega_n^2 = \frac{k}{m} = \frac{20}{m}, \quad 2\zeta\omega_n = \frac{b}{m}.$$

From the plot

$$\begin{aligned} t_r &= 1\,\text{sec} = \frac{1.8}{\omega_n} \Rightarrow \omega_n = 1.8 = \sqrt{\frac{20}{m}} \Rightarrow m = 6.17, \\ M_p &= \frac{y(t_p) - y(\infty)}{y(\infty)} = (0.113 - 0.1)/0.1 \times 100\% = 13.1\% \Rightarrow \zeta = 0.543, \\ b &= 2\zeta\omega_n m = 2(0.543)(1.8)(6.17) = 12.06. \end{aligned}$$

35. Solution:

$$J_m\ddot{\theta}_m + \left(b + \frac{K_tK_e}{R_a}\right)\dot{\theta}_m = \frac{K_t}{R_a}v_a.$$

(a)

$$\begin{aligned} J_m\ddot{\theta}_m s^2 + \left(b + \frac{K_tK_e}{R_a}\right)\dot{\theta}_m s &= \frac{K_t}{R_a}V_a(s) \\ \frac{s\theta_m(s)}{V_a(s)} &= \frac{\frac{K_t}{R_aJ_m}}{s + \frac{b}{J_m} + \frac{K_tK_e}{R_aJ_m}}. \end{aligned}$$

$$\begin{aligned} J_m &= 10 \text{ kg}\cdot\text{m}^2, \\ b &= 1 \text{ N}\cdot\text{m}\cdot\text{sec}, \\ K_e &= 2 \text{ V}\cdot\text{sec}, \\ K_t &= 2 \text{ N}\cdot\text{m/A}, \\ R_a &= 10\Omega\ . \end{aligned}$$

$$\frac{s\theta_m(s)}{V_a(s)} = \frac{0.02}{s + 0.14}.$$

(b) Final Value Theorem

$$\dot{\theta}(\infty) = \frac{s(10)(0.02)}{s(s+\ 0.14)}|_{s=0} = \frac{0.2}{0.14} = 1.42.$$

(c)

$$\frac{\theta_m(s)}{V_a(s)} = \frac{0.02}{s(s + 0.14)}.$$

(d)

$$\begin{aligned} \theta_m(s) &= \frac{0.02K(\theta_r - \theta_m)}{s(s + 0.14)}. \\ \frac{\theta_m(s)}{\theta_r(s)} &= \frac{0.02K}{s^2 + 0.14s + 0.02K}. \end{aligned}$$

(e)

$$\begin{aligned}
M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.2 \qquad (20\%), \\
\zeta &= 0.4559. \\
Y(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \\
2\zeta\omega_n &= 0.14, \\
\omega_n &= \frac{0.14}{2(0.4559)} = 0.15 \text{ rad/ sec}, \\
{\omega^2}_n &= 0.02K, \\
&0 < K < 1.2.
\end{aligned}$$

(f)

$$\omega_n = \frac{1.8}{t_r} = \frac{1.8}{4} = 0.45$$

$$\omega_n^2 = 0.02K \quad \Rightarrow \quad K \geq \frac{(0.45)^2}{0.02} = 10.12$$

36. Solution:

$$J\ddot{\theta} + B\dot{\theta} = T_c$$

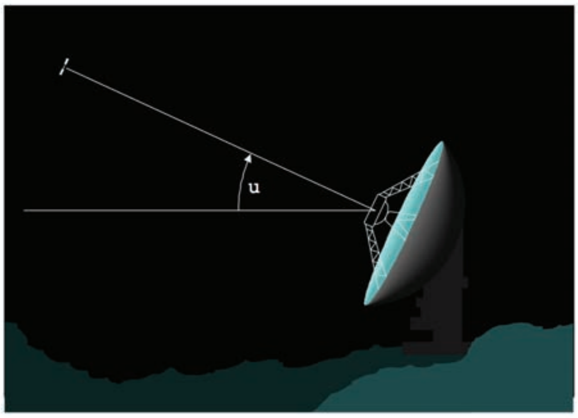

Figure 3.61: Schematic of antenna for Problem 3.36

(a)

$$\begin{aligned} J\Theta s^2 + B\Theta s &= T_c(s), \\ \frac{\Theta(s)}{T_c(s)} &= \frac{1}{s(Js+B)}, \\ J &= 600,000\ \mathrm{kg\cdot m^2}, \\ B &= 20,000\ \mathrm{N\cdot m\cdot sec}, \\ \frac{\Theta(s)}{T_c(s)} &= \frac{1.667\times 10^{-6}}{s(s+\frac{1}{30})}. \end{aligned}$$

(b)

$$\begin{aligned} \Theta(s) &= \frac{1.667\times 10^{-6} K(\Theta_r - \Theta)}{s(s+\frac{1}{30})}, \\ \frac{\Theta(s)}{\Theta_r(s)} &= \frac{1.667K\times 10^{-6}}{s^2+\frac{1}{30}s+1.667K\times 10^{-6}}. \end{aligned}$$

(c)

$$\begin{aligned} M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.1 \qquad (10\%), \\ \zeta &= 0.591. \\ Y(s) &= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}, \\ 2\zeta\omega_n &= \frac{1}{30}, \\ \omega_n &= \frac{\frac{1}{30}}{2(0.591)} = 0.0282\ \mathrm{rad/sec}, \\ \omega_n^2 &= 1.667K\times 10^{-6}, \\ K &< 477. \end{aligned}$$

(d)

$$\begin{aligned} \omega_n &\geq \frac{1.8}{t_r}, \\ \omega_n^2 &= 1.667K\times 10^{-6}, \\ K &\geq 304. \end{aligned}$$

55. Solution

The characteristic equation is $1 + G(s)H(s) = 0$:

$$1 + \frac{k(s+2)}{s(s-1)} = 0 \quad \Rightarrow \quad s^2 + (k-1)s + 2k = 0$$

$$\Rightarrow 2\zeta\omega_n = k - 1 \qquad \text{and} \qquad \omega_n^2 = 2k$$

a) Given $\zeta = 0.5$, we have $(k-1)^2 = 2k$ which leads to $k = 2 \pm \sqrt{3}$. The solution $k = 2 - \sqrt{3}$ is not acceptable because it leads to an unstable system as it is shown by the Routh array:

$$\begin{array}{c|cc} 2 & 1 & 2k \\ 1 & k-1 & \\ 0 & 2k & \end{array}$$

So for the closed-loop stability $k - 1 > 0$. Therefore, the acceptable stabilizing solution is $k = 2 + \sqrt{3} = 3.73$.

b) Based on the Routh Array, at $k = 1$, the system has two poles on the imaginary axis as follows:

$$s^2 + 2 = 0 \quad \Rightarrow \quad s = \pm j\sqrt{2} \quad \Rightarrow \quad \omega_n = \sqrt{2} \quad \text{rad/s}$$

58. Solution:

(a) The characteristic equation is,

$$s(s+1) + Ae^{-Ts} = 0$$

(b) Using $e^{-Ts} \cong 1 - Ts$, the characteristic equation is,

$$s^2 + (1 - TA)s + A = 0$$

The Routh's array is,

$$\begin{array}{lcll} s^2 & : & 1 & A \\ s^1 & : & 1 - TA & 0 \\ s^0 & : & A & \end{array}$$

For stability we must have $A > 0$ and $TA < 1$.
Using $e^{-Ts} \cong \frac{(1-\frac{T}{2}s)}{(1+\frac{T}{2}s)}$, the characteristic equation is,

$$s^3 + \left(1 + \frac{2}{T}\right)s^2 + \left(\frac{2}{T} - A\right)s + \frac{2}{T}A = 0$$

The Routh's array is,

$$\begin{array}{lcll} s^3 & : & 1 & \left(\frac{2}{T} - A\right) \\ s^2 & : & \left(1 + \frac{2}{T}\right) & \frac{2A}{T} \\ s^1 & : & \frac{\left(1+\frac{2}{T}\right)\left(\frac{2}{T} - A\right) - \frac{2A}{T}}{\left(1+\frac{2}{T}\right)} & 0 \\ s^0 & : & \frac{2A}{T} & \end{array}$$

For stability we must have all the coefficients in the first column be positive.

62. Solution

Using the feedback rules straightforwardly gives the following solutions:

$$\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)}$$
$$\frac{Y(s)}{T_d(s)} = \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}$$

63. Solution

Let's name Y_1 the output of $\frac{10}{s+1}$. Therefore, we have:

$$Y_1(s) = \frac{10}{s+1}[R(s) - 2Y_1(s) - Y(s)]$$
$$Y(s) = \frac{1}{s}Y_1(s)$$

Replacing $Y_1(s) = sY(s)$ in the first equation gives:

$$sY(s) = \frac{10}{s+1}[R(s) - 2sY(s) - Y(s)] = \frac{10}{s+1}R(s) - \frac{10(2s+1)}{s+1}Y(s)$$

$$\frac{s^2+s+20s+10}{s+1}Y(s) = \frac{10R(s)}{s+1}$$

which leads to:

$$\frac{Y(s)}{R(s)} = \frac{10}{s^2+21s+10}$$

An alternative is to move the feedback point from $Y_1(s)$ to $Y(s)$. Then we will have a new feedback block as $(2s+1)$. So the transfer function can be computed by the feedback rule as:

$$\frac{Y(s)}{R(s)} = \frac{\frac{10}{s+1}\frac{1}{s}}{1+\frac{10}{s+1}\frac{1}{s}(2s+1)} = \frac{10}{s^2+21s+10}$$

64. Solution:

In order to compute the transfer function, the following steps can be done:

1. Move the feedback point from $A(s)$ to $Z(s)$, then H_3 will be converted to H_3/G_2.

2. Add H_2 and H_3/G_2 and compute the transfer function between W and Z using the feedback rule:

$$G_3(s) = \frac{Z(s)}{W(s)} = \frac{G_1G_2}{1 + G_1G_2\left(H_2 + \frac{H_3}{G_2}\right)}$$

3. Compute the transfer function between $E(s)$ and $Z(s)$ as:

$$G_4(s) = \frac{Z(s)}{E(s)} = K\frac{G_3(s)}{1 + G_3(s)H_1(s)} = \frac{KG_1G_2}{1 + G_1G_2(H_1 + H_2) + G_1H_3}$$

4. Compute the transfer function between $R(s)$ and $Y(s)$, using the feedback rule:

$$\frac{Y(s)}{R(s)} = \frac{G_4/s}{1 + G_4/s} = \frac{KG_1G_2/s}{1 + G_1G_2(H_1 + H_2) + G_1H_3 + KG_1G_2/s}$$

65. Solution:

a) In the first step, the input of block K is moved to Y, which changes the block to $K(s+2)$. In the second step, the inside closed-loop system including the block $1/(s+2)$ and the block 3 in positive feedback is simplified to $\frac{1/(s+2)}{1-3/(s+2)} = \frac{1}{s-1}$. Then in the final step, $T(s) = Y(s)/R(s)$ is computed as follows:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\frac{1}{s(s+10)}\frac{1}{s-1}}{1 + \frac{K(s+2)}{s(s+10)(s-1)}} = \frac{1}{s(s+10)(s-1) + K(s+2)}$$

b) The closed-loop poles are the roots of the characteristic polynomial:

$$s(s+10)(s-1) + K(s+2) = s^3 + 9s^2 + (K-10)s + 2K = 0$$

The Routh array is :

$$\begin{array}{c|cc} 3 & 1 & K-10 \\ 2 & 9 & 2K \\ 1 & -\frac{2K-9(K-10)}{9} & 0 \\ 0 & 2K & \end{array}$$

The closed-loop system is stable if

$$9(K-10) - 2K > 0 \quad \Rightarrow \quad K > 90/7 = 12.85$$