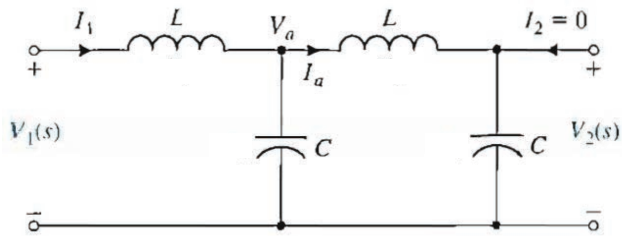
Solutions of Written Exercise 1

Control Systems

Solution of Exercise 1:

Compute the transfer function of the system :



Kirchhoff's voltage law :

$$v_1(t) - v_a(t) = L\frac{di_1(t)}{dt}$$
$$v_a(t) - v_2(t) = L\frac{di_a(t)}{dt}$$

Kirchhoff's current law at V_a and V_2:

$$i_1(t) = i_a(t) + C\frac{dv_a(t)}{dt}$$
$$i_a(t) = C\frac{dv_2(t)}{dt}$$

Taking the Laplace transform gives:

$$V_1(s) - V_a(s) = LsI_1(s)$$
$$V_a(s) - V_2(s) = LsI_a(s)$$
$$I_1(s) = I_a(s) + CsV_a(s)$$
$$I_a(s) = CsV_2(s)$$

Eliminating $I_1(s)$ and $I_a(s)$ gives:

$$V_1(s) - V_a(s) = LCs^2[V_a(s) + V_2(s)]$$
$$V_a(s) - V_2(s) = LCs^2 V_2(s)$$

Then eliminating $V_a(s)$ gives the transfer function:

$$\frac{V_2(s)}{V_1(s)} = \frac{1}{1 + 3LCs^2 + L^2C^2s^4}$$

Solution of Exercise 2:

Consider the application of Newton's law ($\sum F = m\ddot{x}$). From the mass m_v we obtain

$$m_v\ddot{x}_1 = F - k_1(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2).$$

Taking the Laplace transform, and solving for $X_1(s)$ yields

$$X_1(s) = \frac{1}{\Delta_1(s)}F(s) + \frac{b_1 s + k_1}{\Delta_1(s)}X_2(s),$$

where

$$\Delta_1 := m_v s^2 + b_1 s + k_1.$$

From the mass m_t we obtain

$$m_t\ddot{x}_2 = -k_2x_2 - b_2\dot{x}_2 + k_1(x_1 - x_2) + b_1(\dot{x}_1 - \dot{x}_2).$$

Taking the Laplace transform, and solving for $X_2(s)$ yields

$$X_2(s) = \frac{b_1 s + k_1}{\Delta_2(s)}X_1(s),$$

where

$$\Delta_2 := m_t s^2 + (b_1 + b_2)s + k_1 + k_2.$$

Substituting $X_2(s)$ above into the relationship fpr $X_1(s)$ yields the transfer function

$$\frac{X_1(s)}{F(s)} = \frac{\Delta_2(s)}{\Delta_1(s)\Delta_2(s) - (b_1 s + k_1)^2}.$$

Solution of Exercise 3:

The Kirchhoff's voltage law for the DC motor gives:

$$u_m(t) = R_m i_m(t) + K_m\dot{\theta}(t)$$

The electrical generated torque is equal to the mechanical torque:

$$J\ddot{\theta}(t) = K_m i_m(t) = K_m\frac{u_m(t) - K_m\dot{\theta}(t)}{R_m}$$

Taking the Laplace transform from the above equations gives:

$$JR_m s^2\Theta(s) = K_m U_m(s) - K_m^2 s\Theta(s)$$

Which leads to

$$\Theta(s) = \frac{K_m}{s(JR_m s + K_m^2)} U_m(s)$$

On the other hand the dynamic equation of the tank is:

$$A\dot{h}(t) = \alpha\theta(t) - \beta h(t)$$

Taking the Laplace transform gives:

$$AsH(s) = \alpha\Theta(s) - \beta H(s)$$

Therefore:

$$Q(s) = \beta H(s) = \frac{\alpha\beta}{As + \beta}\Theta(s)$$

So the the transfer function between the motor voltage and the output flow rate is:

$$\frac{Q(s)}{U_m(s)} = \frac{\alpha\beta K_m}{s(As+\beta)(JR_m s + K_m^2)}$$

Solution of Exercise 4:

Assume the motor torque is proportional to the input current

$$T_m = ki \ .$$

Then, the equation of motion of the beam is

$$J\ddot{\phi} = ki \ ,$$

where J is the moment of inertia of the beam and shaft (neglecting the inertia of the ball). We assume that forces acting on the ball are due to gravity and friction. Hence, the motion of the ball is described by

$$m\ddot{x} = mg\phi - b\dot{x}$$

where m is the mass of the ball, b is the coefficient of friction, and we have assumed small angles, so that $\sin\phi \approx \phi$. Taking the Laplace transfor of both equations of motion and solving for $X(s)$ yields

$$X(s)/I(s) = \frac{gk/J}{s^2(s^2 + b/m)} \ .$$

Note: The above solution is wrong. The correct response is:

$$\frac{X(s)}{I(s)} = \frac{gk/J}{s^3(s + b/m)}$$

Solution of Exercise 5:

For an ideal op-amp, at node a we have

$$\frac{v_{in} - v_a}{R_1} + \frac{v_o - v_a}{R_1} = 0 \ ,$$

and at node b

$$\frac{v_{in} - v_b}{R_2} = C\dot{v}_b \ ,$$

from it follows that

$$\left[\frac{1}{R_2} + Cs\right] V_b = \frac{1}{R_2} V_{in} \ .$$

Also, for an ideal op-amp, $V_b - V_a = 0$. Then solving for V_b in the above equation and substituting the result into the node a equation for V_a yields

$$\frac{V_o}{V_{in}} = \frac{2}{\frac{1}{R_2} + Cs}\left[\frac{1}{R_2} - \frac{\frac{1}{R_2} + Cs}{2}\right]$$

or

$$\frac{V_o(s)}{V_{in}(s)} = -\frac{R_2 Cs - 1}{R_2 Cs + 1} \ .$$

Solution of Exercise 6:

(a) The equations of motion for the two masses are

$$ML^2\ddot{\theta}_1 + MgL\theta_1 + k\left(\frac{L}{2}\right)^2(\theta_1 - \theta_2) = \frac{L}{2}f(t)$$
$$ML^2\ddot{\theta}_2 + MgL\theta_2 + k\left(\frac{L}{2}\right)^2(\theta_2 - \theta_1) = 0 \ .$$

With $\dot{\theta}_1 = \omega_1$ and $\dot{\theta}_2 = \omega_2$, we have

$$\dot{\omega}_1 = -\left(\frac{g}{L} + \frac{k}{4M}\right)\theta_1 + \frac{k}{4M}\theta_2 + \frac{f(t)}{2ML}$$
$$\dot{\omega}_2 = \frac{k}{4M}\theta_1 - \left(\frac{g}{L} + \frac{k}{4M}\right)\theta_2 \ .$$

(b) Define $a = g/L + k/4M$ and $b = k/4M$. Then

$$\frac{\theta_1(s)}{F(s)} = \frac{1}{2ML}\frac{s^2 + a}{(s^2 + a)^2 - b^2} \ .$$

Solution of Exercise 7:

1. The nonlinear equations of the system are given by:

$$\begin{aligned} v(t) &= L\frac{di(t)}{dt} + Ri(t) \\ mg - k\left(\frac{i(t)}{x(t)}\right)^2 &= m\frac{d^2x(t)}{dt^2} \end{aligned}$$

2. The second term of the above equation is nonlinear and can be linearized by a Taylor series approximation:

$$\begin{aligned} f(t) &\approx f_0 + \left.\frac{df(t)}{di}\right|_{i=i_0,x=x_0}(i(t)-i_0) + \left.\frac{df(t)}{dx}\right|_{i=i_0,x=x_0}(x(t)-x_0) \\ &\approx k\left(\frac{i_0}{x_0}\right)^2 + \frac{2ki_0}{x_0^2}(i(t)-i_0) + \frac{-2ki_0^2}{x_0^3}(x(t)-x_0) \end{aligned}$$

   Then we define new variables $\Delta i(t) = i(t) - i_0$, $\Delta x(t) = x(t) - x_0$ and $\Delta v(t) = v(t) - v_0$. With these new variables, the first equation becomes:

$$\Delta v(t) + v_0 = L\frac{d\Delta i(t)}{dt} + R\Delta i(t) + Ri_0$$

   Since in equilibrium, we have $v_0 = Ri_0$, we obtain:

$$\Delta v(t) = L\frac{d\Delta i(t)}{dt} + R\Delta i(t)$$

   In the same way, for the second equation we have:

$$mg - k\left(\frac{i_0}{x_0}\right)^2 - \frac{2ki_0}{x_0^2}\Delta i(t) - \frac{-2ki_0^2}{x_0^3}\Delta x(t) = m\frac{d^2\Delta x(t)}{dt^2}$$

   In equilibrium we have $mg = ki_0^2/x_0^2$ which leads to the following linear equation:

$$-\frac{2ki_0}{x_0^2}\Delta i(t) + \frac{2ki_0^2}{x_0^3}\Delta x(t) = m\frac{d^2\Delta x(t)}{dt^2}$$

3. Let's define the Laplace transform of $\Delta i(t), \Delta x(t)$ and $\Delta v(t)$ respectively $I(s), X(s)$ and $V(s)$. Therefore, by taking the Laplace transform of the linearized equations, we obtain:

$$\begin{aligned} &V(s) = LsI(s) + RI(s) \\ &-\frac{2ki_0}{x_0^2}I(s) + \frac{2ki_0^2}{x_0^3}X(s) = ms^2X(s) \end{aligned}$$

   From the first equation we compute $I(s) = \frac{V(s)}{Ls+R}$ and replace it in the second equation to find:

$$\frac{-2ki_0V(s)}{x_0^2(Ls+R)} = \left(ms^2 - \frac{2ki_0^2}{x_0^3}\right)X(s) \quad \Rightarrow \quad \frac{X(s)}{V(s)} = \frac{-2ki_0x_0}{(Ls+R)(x_0^3ms^2 - 2ki_0^2)}$$