Convolution

Exercise

The impulse response of a system is $h(t) = 6e^{-3t}$ for $t \geq 0$. Compute the response of the system to $u(t)$ defined as :

$$u(t) = \begin{cases} 0 & t < 0, t > 4 \\ 2 & 0 \leq t \leq 4 \end{cases}$$

(A) $y(t) = [-4e^{-3t} + 4e^{-3(t-4)}]\mathbf{1}(t)$

(B) $y(t) = [4(1 - e^{-3t})]\mathbf{1}(t)$

(C) $y(t) = 4(1 - e^{-3t})\mathbf{1}(t) - 4(1 - e^{-3(t-4)})\mathbf{1}(t-4)$

(D) $y(t) = [2(1 - e^{-3t}) - 2(1 - e^{-3(t-4)})]\mathbf{1}(t)$

(E) I do not know

Convolution

Solution (by superposition principle)

Note that $u(t) = 2\mathbf{1}(t) - 2\mathbf{1}(t-4)$. So we can compute first the response of the system to a unit step response $\mathbf{1}(t)$:

$$y_1(t) = \int_0^t 6e^{-3t+3\tau}\mathbf{1}(\tau)d\tau = 2e^{-3t+3\tau}\Big|_{\tau=0}^{\tau=t}\mathbf{1}(t) = 2(1-e^{-3t})\mathbf{1}(t)$$

Therefore, based on the superposition principle :

$$y(t) = 2y_1(t) - 2y_1(t-4) = 4(1-e^{-3t})\mathbf{1}(t) - 4(1-e^{-3(t-4)})\mathbf{1}(t-4)$$

Convolution

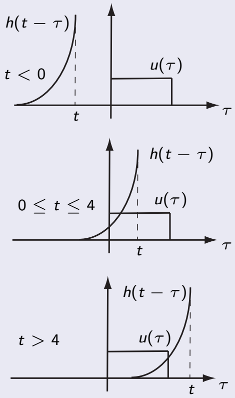
Solution (by integration)

- For $t < 0$ we have $y(t) = 0$.
- For $0 \le t \le 4$ we have :

$$y(t) = \int_0^t 2(6e^{-3t+3\tau})d\tau = 4(1 - e^{-3t})$$

- For $4 < t$ we have :

$$y(t) = \int_0^4 2(6e^{-3t+3\tau})d\tau = 4e^{-3(t-4)} - 4e^{-3t}$$



Therefore, $y(t)$ for all t is :

$$\begin{aligned} y(t) &= 4(1 - e^{-3t})[\mathbf{1}(t) - \mathbf{1}(t-4)] + [4e^{-3(t-4)} - 4e^{-3t}]\mathbf{1}(t-4) \\ &= 4(1 - e^{-3t})\mathbf{1}(t) - 4(1 - e^{-3(t-4)})\mathbf{1}(t-4) \end{aligned}$$

Routh Criterion

Exercise

Determine whether any of the roots of the following polynomial are in the RHP :

$$A(s) = s^3 + 7s^2 + 25s + 35$$

(A) One RHP root (B) No RHP root

(C) Two RHP roots (D) Three RHP roots

Solution

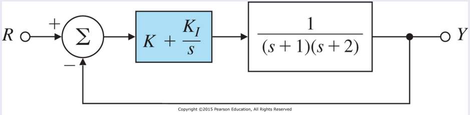
$$\begin{array}{c|cc}
3 & 1 & 25 \\
2 & 7 & 35 \\
1 & -\frac{35-175}{7} = 20 & 0 \\
0 & -\frac{0-35(20)}{20} = 35 &
\end{array}$$

There is no change of sign in the first column. The polynomial has no RHP root (B).

Routh Criterion

Exercise

Determine the range of (K, K_I) over which this closed-loop system is stable.



(A) $K > \frac{1}{3}K_I - 2$ and $K_I > 0$

(B) $K > \frac{1}{2}K_I - 3$ and $K_I > 0$

(C) $K > \frac{1}{3}K_I - 1$ and $K_I > 1$

(D) None of the above

(E) I do not know

Routh Criterion

Solution

First we compute the transfer function between r and y :

$$\frac{Y(s)}{R(s)} = \frac{\frac{K+K_I/s}{(s+1)(s+2)}}{1+\frac{K+K_I/s}{(s+1)(s+2)}} = \frac{Ks+K_I}{s^3+3s^2+(2+K)s+K_I}$$

The closed-loop system is stable if all poles are in the LHP.

$$\begin{array}{c|cc} 3 & 1 & 2+K \\ 2 & 3 & K_I \\ 1 & -\frac{K_I-6-3K}{3} & 0 \\ 0 & K_I & \end{array}$$

The closed-loop system is stable if $K > \frac{1}{3}K_I - 2$ and $K_I > 0$

Steady-State Errors

Question : Exam 2015

Consider $G(s) = \dfrac{s+100}{s(s+2)(s+30)}$, with a proportional controller $D_c(s) = 10$, compute the steady-state errors for tracking unit step and ramp reference signals.

(A) ss error for step $= 0$, ss error for ramp $= \infty$

(B) ss error for step $= 0$, ss error for ramp $= 16.67$

(C) ss error for step $= 0$, ss error for ramp $= 0.06$

(D) ss error for step $= 0.06$, ss error for ramp $= \infty$

Solution : The transfer function $G(s)D_c(s)$ has one integrator, then the steady-state error for a unit step is zero and

$$K_v = \lim_{s \to 0} s \frac{10(s+100)}{s^3 + 32s^2 + 60s} = 16.67$$

Therefore, the steady-state tracking error for a unit ramp $= 1/K_v = 0.06$

Steady-state Error (Regulation Performance)

Question

Given the following controller and plant model :

$$D_c(s) = 10 \qquad ; \qquad G(s) = \frac{(s+100)}{s(s+2)}$$

The steady-state error for a step and ramp disturbance is respectively :

A) -0.002, $-\infty$ B) -0.1, $-\infty$ C) 0, -0.1 D) ∞, -0.002

Solution

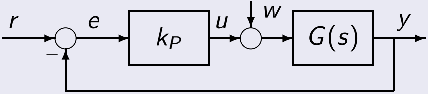
For step disturbance $W(s) = 1/s$ and :

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s \frac{-G(s)}{1+G(s)D_c(s)} \frac{1}{s} = \lim_{s\to 0} \frac{-(s+100)}{s(s+2)+10(s+100)} = -0.1$$

For ramp disturbance $W(s) = 1/s^2$ and :

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s \frac{-G(s)}{1+G(s)D_c(s)} \frac{1}{s^2} = \lim_{s\to 0} \frac{-(s+100)}{s^2(s+2)+10s(s+100)} = -\infty$$

Proportional Controller



Control signal is proportional to the error signal.

$$u(t) = k_P e(t) \quad \Rightarrow U(s) = k_P E(s)$$

In order to have a non-zero control signal we SHOULD have a tracking error !

Question : Tracking a unit step signal

Assume that $r(t) = 1$ and $R(s) = 1/s$ (no disturbance $w(t) = 0$). Find the steady-state tracking error :

A : 0 B : $\dfrac{k_P G(0)}{1 + k_P G(0)}$

C : $\dfrac{1}{1 + k_P G(0)}$ D : $\dfrac{k_P}{1 + k_P G(0)}$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1 + k_P G(s)}\frac{1}{s} = \frac{1}{1 + k_P G(0)}$$

Ziegler-Nichols Tuning Methods

Exercise

La relation entre la température du liquide $y(t)$ dans une cuve du mélange et la température du liquide d'alimentation $u(t)$ est donnée par :

$$\frac{dy(t)}{dt} + 2y(t) = 2u(t-0.3)$$

1. A partir de la réponse indicielle du système, dimensionner un régulateur PI par la première méthode de Ziegler-Nichols.

A : $K_p = 0.5, T_i = 0.5$ B : $K_p = 0.66, T_i = 1$

C : $K_p = 1.5, T_i = 1$ D : None of the above

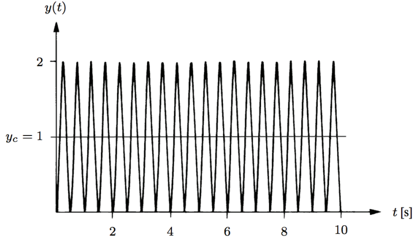
Solution : $G(s) = \dfrac{2e^{-0.3s}}{s+2} \quad \Rightarrow \quad L = 0.3 \quad \text{et} \quad R = 2$

$$\Rightarrow \quad K_p = 0.9/RL = 0.9/0.6 = 1.5 \quad \text{et} \quad T_i = L/0.3 = 1$$

Ziegler-Nichols Tuning Methods

Exercise

1.11.26 Un processus est commandé en boucle fermée par un régulateur proportionnel. Avec un gain valant 40,4, la réponse indicielle obtenue apparaît dans la figure 1.96.



Compute a PID controller based on the second method of ZN.

Solution :

$$K_u = 40.4 \quad , \quad P_u = 0.5$$

$$K_p = 0.6K_u = 24.24 \quad , \quad T_i = 0.5P_u = 0.25 \quad , \quad T_d = 0.125P_u = 0.0625$$

Exercise (Exam-2015)

La relation entre la température du liquide $y(t)$ dans une cuve du mélange et la température du liquide d'alimentation $u(t)$ est donnée par :

$$\frac{dy(t)}{dt} + 2y(t) = 2u(t - 0.3)$$

1. Dimensionner un régulateur PI par la méthode du modèle à poursuivre (*Model Reference Control*) pour obtenir une bande passante de 2.5 rad/s en boucle fermée.

A : $k_P = 0.357, k_I = 0.5$ B : $k_P = 1.4, k_I = 0.7$

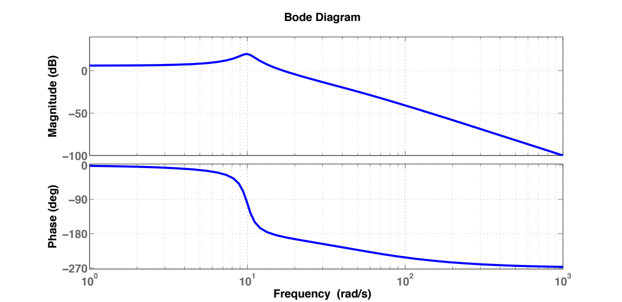
C : $k_P = 0.714, k_I = 1.428$ D : None of the above

$$G(s) = \frac{2e^{-0.3s}}{s+2} \quad \Rightarrow \quad \tau = 0.5, \quad \gamma = 1, \quad \theta = 0.3, \quad \tau_m = 1/2.5 = 0.4$$

$$\Rightarrow \quad k_P = \frac{\tau}{\gamma(\tau_m + \theta)} = \frac{0.5}{0.7} = 0.714, \quad k_I = \frac{1}{0.7} = 1.428$$

Some Information in the Bode Plots

Exercise



Question : How many poles ? How many zeros ?

A) 2 poles, no zero

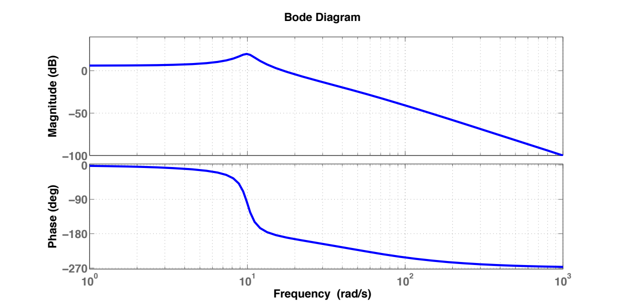
B) 3 poles, one zero

C) 3 poles, no zero

D) 4 poles, one zero

Some Information in the Bode Plots

Exercise

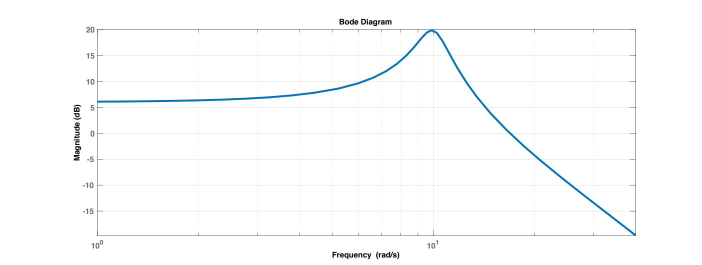


Question : Frequency of poles ?

A) 2 real poles around 10 rad/s and one pole around 5 rad/s
B) 2 complex poles around 10 rad/s and one pole around 20 rad/s
C) 2 complex poles around 10 rad/s and one pole around 100 rad/s
D) 2 complex poles around 10 rad/s and one pole around 50 rad/s

Some Information in the Bode Plots

Exercise (zoom on the magnitude plot)



Question 1 : What is the value of the steady-state gain ?

(A) 1 **(B)** 2 **(C)** 6 **(D)** I don't know

Question 2 : What is the bandwidth (rad/s) ?

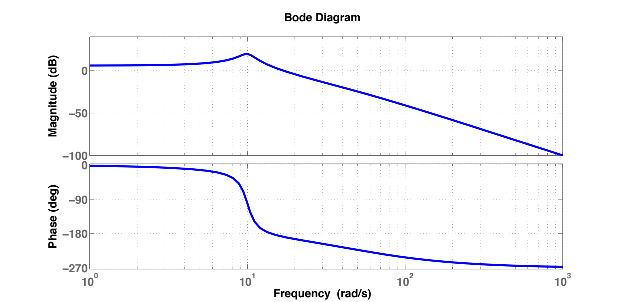
(A) 10 **(B)** 15 **(C)** 20 **(D)** I don't know

Question 3 : What is the damping factor ?

(A) $\zeta = 0.1$ **(B)** $\zeta = 0.3$ **(C)** $\zeta = 0.7$ **(D)** I don't know

Some Information in the Bode Plots

Exercise



Transfer function of the system ?

$$\text{SS gain}=2\ ,\ \omega_n = 10,\ \zeta = 0.1,\ \text{pole}= 50\ (\tau = 1/50 = 0.02)$$

$$G(s) = \frac{2 \times 10^2}{(s^2 + 2 * 0.1 * 10s + 10^2)(0.02s + 1)}$$

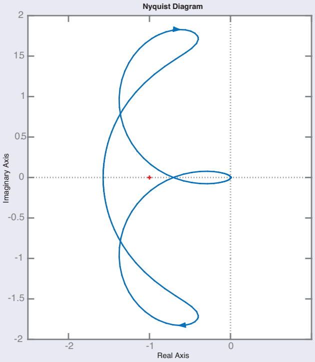
Nyquist Stability Criterion

Exercise

Le diagram Nyquist de

$$G(s) = \frac{0.0026(s-22)(s-2)(s+0.1053)}{(s-0.0952)(s^2+0.2266s+0.0793)}$$

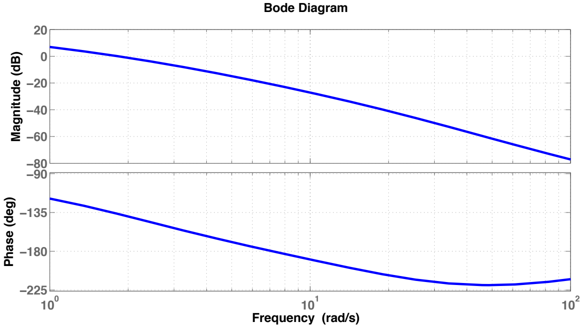
est donnée dans cette figure. Utiliser le Théorème de Nyquist pour étudier la stabilité de ce système en boucle fermée avec un régulateur proportionnel pour les valeurs suivantes :
(a) $k_P = 1$, (b) $k_P = 2$ and (c) $k_P = 0.5$.



Solution : We have $P = 1$. For $k_P = 1$, we have one counterclockwise encirclement so the system is stable. For $k_P = 2$, we have one clockwise encirclement so the system is unstable (2 RHP poles). For $k_P = 0.5$, we have zero encirclement so the system is unstable (one RHP pole).

Exercise

The frequency response of the open-loop transfer function is given by :



Find the following information from the diagram :

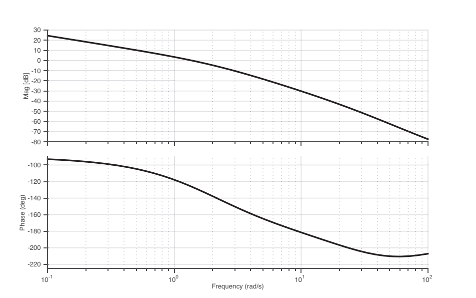
	(A)	(B)	(C)	(D)
ω_{cr}	1.8	7	1	10^4
ω_c	1.8	10	7	45
GM	20dB	-20dB	20	0.1
PM	60°	90°	135°	45°

Stability margins

Question : Exam 2015

Le diagramme de Bode de $G(s) = \dfrac{s+100}{s(s+2)(s+30)}$ est donné. Avec un régulateur proportionnel $D_c(s) = 10$, calculer la marge de gain et la marge de phase approximative (0.2 point).

(A) GM $\approx$ 30dB, PM $\approx 55^\circ$
(B) GM $\approx$ 20dB, PM $\approx 30^\circ$
(C) GM $\approx$ 10dB, PM $\approx 15^\circ$
(D) GM $\approx$ 30dB, PM $\approx 30^\circ$



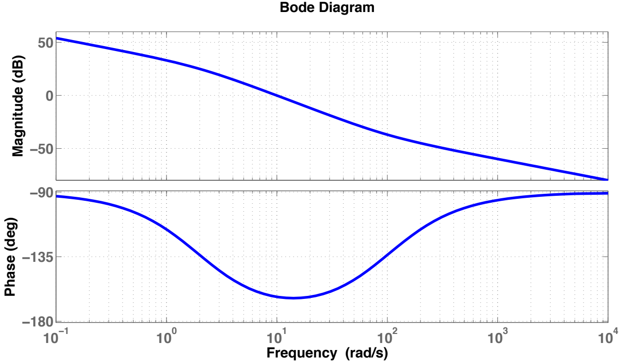
Solution :

$$\omega_{cr} \approx 10 rad/s \Rightarrow |D_c(j\omega_{cr})G(j\omega_{cr})| \approx -30dB + 20dB = -10dB \Rightarrow \quad \text{GM} \approx 10dB$$

$$\omega_c \approx 5.5 rad/s \quad \Rightarrow \arg[D_c(j\omega_c)G(j\omega_c)] = -165^\circ \Rightarrow \quad \text{PM} \approx 15^\circ$$

Exercise

Given $G(s) = \dfrac{(s+100)}{s(s+2)}$; Find a proportional controller k_P such that



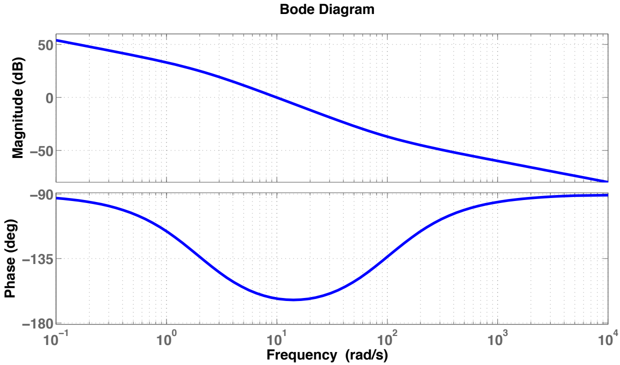
The phase margin is equal to 45° :

A) $k_P = 25$ or $k_P = -37$ **B)** $k_P = -25$ or $k_P = 37$

C) $k_P = 0.056$ or $k_P = 70$ **D)** $k_P = 17.8$ or $k_P = 70$

Exercise

Given $G(s) = \dfrac{(s+100)}{s(s+2)}$; Find a proportional controller k_P such that



The crossover frequency is equal to 30 rad/s :

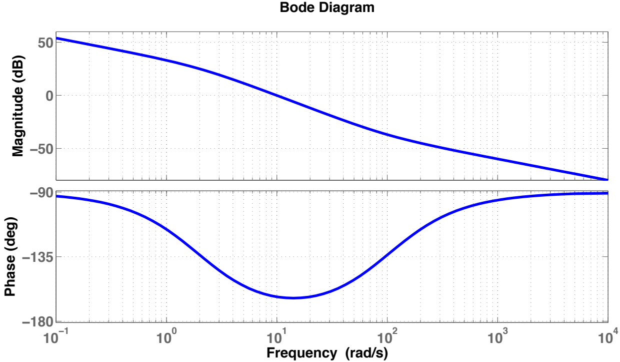
A) $k_P = 20$

B) $k_P = 10$

C) $k_P = 0.1$

D) $k_P = -20$

Given $G(s) = \dfrac{(s+100)}{s(s+2)}$; Find a proportional controller K_p such that



The steady-state error for tracking a ramp signal is 0.01 :

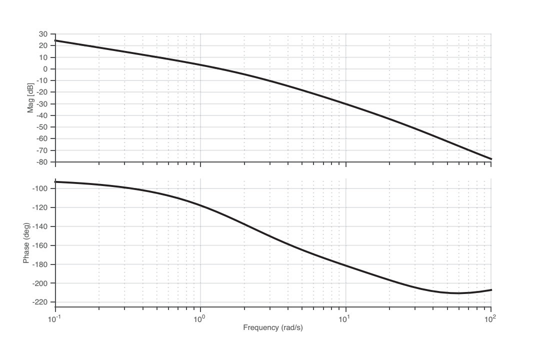
A) $k_P = 2$ **B)** $k_P = 10$

C) $k_P = 20$ **D)** $k_P = 100$

Loop-Shaping Method

Exercise

The Bode diagram of $G(s) = \dfrac{s+100}{s(s+2)(s+30)}$ is given :



Design a PID controller with the Loop-Shaping method to follow a ramp with no steady-state error, obtain a crossover frequency of 15 rad/s and a phase margin of around 60 degrees.

Loop-Shaping Method

Solution

- To eliminate the steady-state error, a PI controller is designed such that its zero at $-1/T_i$ cancel the pole of $G(s)$ at -2, i.e. ($T_i = 1/2$) :

$$D_c(s) = K_p\left(1 + \frac{1}{0.5s}\right) = \frac{K_p}{s}(s+2)$$

- In order to have a slope of -20dB/dec around the crossover frequency, we add a zero well before 15 rad/s
(e.g. $1/T_d = 15/3 = 5 \quad \Rightarrow T_d = 0.2$) :

$$D_c(s) = \frac{K_p}{s}(s+2)(1+0.2s)$$

- Finally we compute K_p to obtain $\omega_c = 15$ rad/s :

$$|D_c(j15)G(j15)| = 1 \quad \Rightarrow \quad K_p = 23.6$$

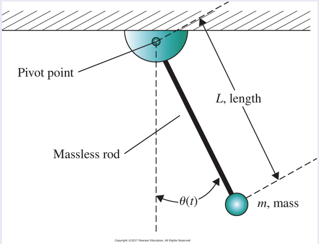
State-Space Models

Question

For a simple pendulum, the nonlinear equation of motion is :

$$\ddot{\theta}(t) + \frac{g}{L}\sin\theta(t) + \frac{k}{m}\dot{\theta}(t) = 0$$

where k is the friction coefficient. Linearize the system about $\theta_0 = 0$. Obtain a state space representation (output is $\theta(t)$).



Solution

The state variables are $\mathbf{x}(t) = \begin{bmatrix} \theta(t) & \omega(t) \end{bmatrix}$.

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{g}{L}x_1(t) - \frac{k}{m}x_2(t)$$

$$y(t) = x_1(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & \frac{-k}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Exercise

Consider the system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The system is :

(A) Controllable, observable

(B) Not Controllable, not observable

(C) Controllable, not observable

(D) Not Controllable, observable

Solution C : $\mathcal{C} = \begin{bmatrix} 0 & 2 \\ 2 & -8 \end{bmatrix}$ $\mathcal{O} = \begin{bmatrix} 0 & 2 \\ 0 & -8 \end{bmatrix}$

State-Space Models

Exercise

For the following state-space model

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t) \qquad y(t) = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \mathbf{x}(t)$$

Find the transfer function model $G(s)$ between $U(s)$ and $Y(s)$.

$$\textbf{(A)}\, G(s) = \frac{5s^2+32s+35}{s^3+9s^2+23s+15} \quad \textbf{(B)}\, G(s) = \frac{5s^2+32s+35}{s^4+9s^2+23s+15}$$

$$\textbf{(C)}\, G(s) = \frac{2s^2+16s+22}{s^3+9s^2+23s+15} \quad \textbf{(D)}\, G(s) = \frac{5s+32}{s^2+32s+9}$$

Solution C :

$$G(s) = \frac{1}{s+1} + \frac{2}{s+3} + \frac{-1}{s+5} = \frac{2s^2+16s+22}{s^3+9s^2+23s+15}$$

State Feedback Controller (Pole Placement)

Exercise

Consider the system : $G(s) = \dfrac{3s^2 + 5s - 5}{s^3 + 12s^2 + 10s + 5}$.
Determine a state feedback controller $\mathbf{K}$ so that the closed-loop poles are -3, -4 and -6.

$$\textbf{(A)}\mathbf{K} = [1 \quad 44 \quad 67] \quad \textbf{(B)}\mathbf{K} = [10 \quad 44 \quad 67]$$
$$\textbf{(C)}\mathbf{K} = [44 \quad 1 \quad 1] \quad \textbf{(D)}\mathbf{K} = [1 \quad 67 \quad 44]$$

Solution : The system can be represented in control canonical form :

$$\mathbf{A} = \begin{bmatrix} -12 & -10 & -5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 3 & 5 & -5 \end{bmatrix}$$

The closed-loop characteristic polynomial is :

$$\alpha_c(s) = (s+3)(s+4)(s+6) = s^3 + 13s^2 + 54s + 72$$

Then $K_1 = -12 + 13 = 1$, $K_2 = -10 + 54 = 44$ and $K_3 = -5 + 72 = 67$.

State Feedback Controller

Exercise

Consider a first order unstable system : $\quad G(s) = \dfrac{3}{s-2}$

1. Give a state-space representation of the system.

   (A) $A=-2, B=3, C=1$ **(B)** $A=2, B=1, C=3$
   (C) $A=2, B=1, C=-3$ **(D)** $A=-2, B=-1, C=3$

2. Compute an LQR controller that minimizes $\mathcal{J} = \int_0^\infty [y^2(t) + u^2(t)]dt$.

   (A) $K = 2 \pm \sqrt{13}$ **(B)** $K = 2 - \sqrt{13}$
   (C) $K = 2 + \sqrt{13}$ **(D)** None of the above

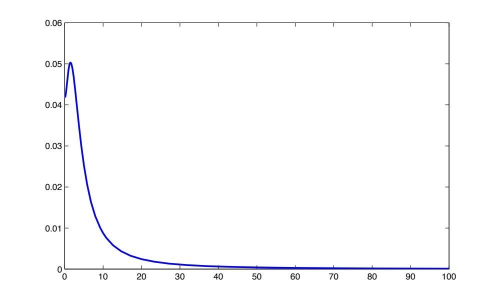
Solution : We have A=2, B=1 and C=3.
The LQR Riccati equation is : $\quad A^T P + PA - PBR^{-1}B^T P + Q = 0$ with $R = 1$ and $Q = C^T C = 9 \quad \Rightarrow \quad 2P + 2P - P^2 + 9 = 0 \quad \Rightarrow \quad P = 2 \pm \sqrt{13}$.
Only the positive solution $P = 2 + \sqrt{13}$ is acceptable that gives
$K = R^{-1}B^T P = 2 + \sqrt{13}$.

Example

Find a sampling period for the control of following system :

$$G(s) = \frac{s+1}{(s+2)(s+3)(s+4)}$$

1. Frequency response in a linear scale : $\omega_e > 2\omega_0$



(A) : $\omega_e \approx 20$ rad/s

(B) : $\omega_e \approx 100$ rad/s

(C) : $\omega_e \approx 200$ rad/s

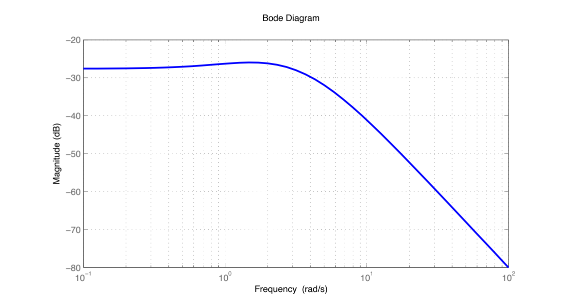
(D) : $\omega_e \approx 1000$ rad/s

$$\omega_0 \approx 40 \text{ rad/s} \quad \Rightarrow \quad \omega_e \approx 100 \text{ rad/s} \quad \Rightarrow \quad h = \frac{2\pi}{100} \approx 0.063s$$

Choice of Sampling Period

Example

2. Bode diagram : $\omega_e = (20 \textbf{ to } 50) \times$ **Bandwidth**



(A) : $\omega_e \approx (10 \text{ to } 50)$ rad/s
(B) : $\omega_e \approx (20 \text{ to } 100)$ rad/s
(C) : $\omega_e \approx (80 \text{ to } 200)$ rad/s
(D) : $\omega_e \approx (120 \text{ to } 300)$ rad/s

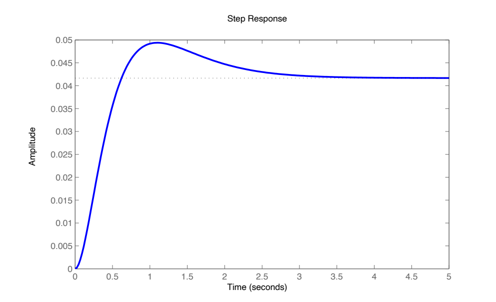
$$\text{Bandwidth} = 4.3978\text{rad/s} \quad \Rightarrow \quad \omega_e = (20 \text{ to } 50) \times 4.4 = (88 \text{ to } 220)\text{rad/s}$$

$$\Rightarrow \quad f_e = \frac{(88 \text{ to } 220)}{2\pi} = (14 \text{ to } 35) \text{ Hz} \quad \Rightarrow \quad h = \frac{1}{f_e} = (0.028 \text{ to } 0.071)s$$

Choice of Sampling Period

Example

3. Step Response : (5 to 10) samples during the rise time.



(A) : $h \approx (0.05 \text{ to } 0.1)$ s

(B) : $h \approx (0.01 \text{ to } 0.02)$ s

(C) : $h \approx (0.2 \text{ to } 0.4)$ s

(D) : $h \approx (0.5 \text{ to } 1)$ s

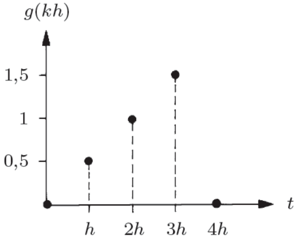
$$\text{Rise-Time} : 0.4286s \quad \Rightarrow \quad h \approx (0.043 \text{ to } 0.086)s$$

$$\Rightarrow \quad \omega_e = \frac{2\pi}{h} = (73 \text{ to } 146) \text{ rad/s}$$

Discrete-Time Systems

Question

The impulse response $\{g(kh)\}$ of an LTI system is given. Compute the response of the system when excited by the signal $\{\mathbf{1}, 1, 1, 1, 0, 0, ...\}$.



(A) $\{y(kh)\} = \{\mathbf{0}, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, \ldots, \}$
(B) $\{y(kh)\} = \{\mathbf{0}, 0.5, 1.5, 2.5, 3, 2.5, 1.5, 0, 0, \ldots, \}$
(C) $\{y(kh)\} = \{\mathbf{0}, 0.5, 1.5, 3, 3, 2.5, 1.5, 0, 0, \ldots, \}$
(D) $\{y(kh)\} = \{\mathbf{0}, 0.5, 1, 1.5, 0.5, 1, 1.5, 0.5, 1, \ldots, \}$

Solution :

$$\{y(kh)\} = \{\mathbf{0}, 0.5, 1.5, 3, 3, 2.5, 1.5, 0, 0, \ldots, \}$$

Properties of the z-Transform

Question

Compute the z-transform of $w(kh) = \cos(\omega kh)$.

(A) $\dfrac{\cos(\omega h)z}{z^2 - 2\sin(\omega h)z + 1}$ (B) $\dfrac{2z^2 - 2\cos(\omega h)z}{z^2 - 2\cos(\omega h)z + 1}$

(C) $\dfrac{z^2 - \cos(\omega h)z}{z^2 - 2\cos(\omega h)z + 1}$ (D) $\dfrac{z^2 + \cos(\omega h)z}{z^2 - 2\sin(\omega h)z + 1}$

$$\mathcal{Z}\{\cos(\omega kh)\} = \mathcal{Z}\left\{\frac{e^{j\omega kh} + e^{-j\omega kh}}{2}\right\} = \frac{1}{2}\left(\mathcal{Z}\{e^{j\omega kh}\} + \mathcal{Z}\{e^{-j\omega kh}\}\right)$$

$$= \frac{1}{2}\left(\frac{z}{z - e^{j\omega h}} + \frac{z}{z - e^{-j\omega h}}\right) = \frac{z^2 - \cos(\omega h)z}{z^2 - 2\cos(\omega h)z + 1}$$

Properties of the z-Transform

Question

Find the final value of $w(k)$ if $W(z) = \dfrac{0.5z}{(z-1)(z-0.5)}$.

$$\begin{array}{llll} \text{(A)} & w(\infty) = -1 & \text{(B)} & w(\infty) = 0 \\ \text{(C)} & w(\infty) = \infty & \text{(D)} & w(\infty) = 1 \end{array}$$

We have only one pole at 1 (the other is inside the unit circle) so

$$\lim_{k\to\infty} w(k) = \lim_{z\to 1}(z-1)\frac{0.5z}{(z-1)(z-0.5)} = 1$$

Properties of the z-Transform

Question

Find the final value of $w(k)$ if $W(z) = \dfrac{z}{z+2}$.

(A)	$w(\infty) = 1/3$	(B)	$w(\infty) = 0$
(C)	$w(\infty) = \infty$	(D)	None of the above

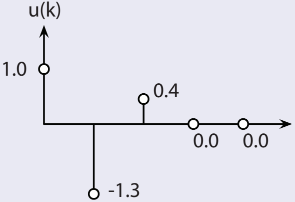
Solution :

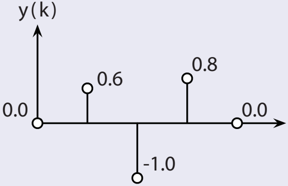
$$\lim_{k\to\infty} w(k) = \lim_{z\to 1}(z-1)\frac{z}{z+2} = 0$$

The result is not correct, because we have one pole outside the unit circle. In fact, the limit does not exist because $W(z) = \mathcal{Z}\{(-2)^k\}$.

Question : Exam 2015

Soit $u(k)$ et $y(k)$, respectivement, l'entrée et la sortie d'un système discret :





1. Trouver la fonction de transfert du système.

(A) $\dfrac{0.6z^2 - z + 0.8}{z(z^2 - 1.3z + 0.4)}$ (B) $\dfrac{0.6z^2 - z + 0.8}{z^2 - 1.3z + 0.4}$

(C) $\dfrac{z^2 - 1.3z + 0.4}{z(0.6z^2 - z + 0.8)}$ (D) None of the above

Inverse z-Transform

Question : Exam 2015

1. Calculer la réponse indicielle (*step response*) du système.

$$
\begin{array}{ll}
\text{(A)} & y(k) = 2 - 8(-0.8)^k + 6(-0.5)^k \\
\text{(B)} & y(k) = 2 - 8(0.8)^k + 6(0.5)^k \\
\text{(C)} & y(k) = -2\Delta(k) + 4 - 8(0.8)^k + 6(0.5)^k \\
\text{(D)} & \text{None of the above}
\end{array}
$$

Solution : Pour la réponse indicielle, $U(z) = \dfrac{z}{z-1}$:

$$Y(z) = \frac{0.6z^2 - z + 0.8}{z(z^2 - 1.3z + 0.4)} \frac{z}{z-1} = c_0 + \frac{c_1 z}{z-1} + \frac{c_2 z}{z-0.8} + \frac{c_3 z}{z-0.5}$$

$$c_0 = \lim_{z\to 0} Y(z) = -2 \qquad c_1 = \lim_{z\to 1} \frac{z-1}{z} Y(z) = 4$$

$$c_2 = \lim_{z\to 0.8} \frac{z-0.8}{z} Y(z) = -8 \qquad c_3 = \lim_{z\to 0.5} \frac{z-0.5}{z} Y(z) = 6$$

$$y(k) = -2\Delta(k) + 4 - 8(0.8)^k + 6(0.5)^k \qquad k \geq 0$$

Inverse z-Transform

Question

Find using numerical inversion the inverse Z transform of

$$W(z) = \frac{z+3}{z^2 - 3z + 2}$$

$$\begin{array}{ll} \text{(A)} & \{w(k)\} = \{\ldots, \mathbf{1}, 6, 16, 36, \ldots\} \\ \text{(B)} & \{w(k)\} = \{\ldots, \mathbf{0}, 1, 0, -2, -6, \ldots\} \\ \text{(C)} & \{w(k)\} = \{\ldots, \mathbf{0}, 1, 6, 16, 26, \ldots\} \\ \text{(D)} & \text{None of the above} \end{array}$$

Note that $a_1 = -3, a_2 = 2, b_0 = 0, b_1 = 1$ and $b_2 = 3$:

$$w(0) = 0 \qquad ; \qquad w(1) = 1 - (-3 \times 0) = 1$$
$$w(2) = 3 - (-3(1) + 2(0)) = 6$$
$$w(3) = 0 - (-3(6) + 2(1)) = 16$$
$$w(4) = 0 - (-3(16) + 2(6)) = 36$$

Zero-Pole Matching Method

Question

Given $G(s) = \dfrac{4}{s(s+2)}$ find $H(z)$ using the zero-pole matching method.

(A) $H(z) = \dfrac{c}{z(z - e^{-2h})}$ **(B)** $H(z) = \dfrac{c(z+1)}{(z-1)(z - e^{-2h})}$

(C) $H(z) = \dfrac{c}{(z - e^{-2h})}$ **(D)** $H(z) = \dfrac{c}{(z-1)(z - e^{-2h})}$

Match the gains at $\omega = 1$.

(A) $c = \infty$ **(B)** $c = (1 - e^{-2h})$

(C) $c = 2(1 - e^{-2h})$ **(D)** $c = \left| \dfrac{4(e^{jh} - 1)(e^{jh} - e^{-2h})}{\sqrt{5}(e^{jh} + 1)} \right|$

Zero-Pole Matching Method

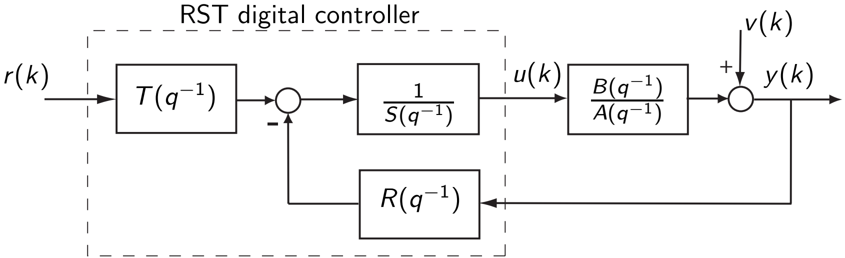
Solution : We have two poles at 1 and e^{-2h} and we add one zero at -1 to have a relative degree of 1. Therefore :

$$H(z) = \frac{c(z+1)}{(z-1)(z-e^{-2h})}$$

To match the gains at $\omega = 1$, we should have $|G(j1)| = |H(e^{jh})|$:

$$\left|\frac{4}{j1(j1+2)}\right| = \left|\frac{c(e^{jh}+1)}{(e^{jh}-1)(e^{jh}-e^{-2h})}\right| \Rightarrow \quad c = \left|\frac{4(e^{jh}-1)(e^{jh}-e^{-2h})}{\sqrt{5}(e^{jh}+1)}\right|$$

Question



- What is the transfer function between r and y assuming $v(k) = 0$

$$\frac{T(q^{-1})B(q^{-1})}{A(q^{-1})S(q^{-1}) + B(q^{-1})R(q^{-1})}$$

- What is the transfer function between v and y assuming $r(k) = 0$

$$\frac{A(q^{-1})S(q^{-1})}{A(q^{-1})S(q^{-1}) + B(q^{-1})R(q^{-1})}$$

Regulation : Computation of R and S

Question

$$\text{Given} \qquad G(q^{-1}) = \frac{0.2q^{-2}}{1 - 0.8q^{-1}}$$

Compute minimum order of $R(q^{-1})$ and $S(q^{-1})$:

(A) $n_R = 1, n_S = 1$ **(B)** $n_R = 0, n_S = 0$

(C) $n_R = 0, n_S = 1$ **(D)** $n_R = 0, n_S = 2$

Solution :

$$n_A = 1, n_B = 2$$

$$n_R = n_A - 1 = 0 \qquad n_S = n_B - 1 = 1$$

Regulation : Computation of R and S

Question

$$\text{Given} \qquad G(q^{-1}) = \frac{0.2q^{-2}}{1 - 0.8q^{-1}}$$

Compute $R(q^{-1}) = r_0$ and $S(q^{-1}) = 1 + s_1 q^{-1}$ to place the closed loop poles at the roots of $P(q^{-1}) = 1 - 1.3q^{-1} + 0.5q^{-2}$.

(A)	$r_0 = 0.5, s_1 = -0.5$	**(B)**	$r_0 = -0.5, s_1 = -0.5$
(C)	$r_0 = -0.5, s_1 = 0.5$	**(D)**	$r_0 = -5.9, s_1 = -2.1$

Solution :

$$(1 - 0.8q^{-1})(1 + s_1 q^{-1}) + 0.2q^{-2} r_0 = 1 - 1.3q^{-1} + 0.5q^{-2}$$

$$\begin{aligned} -0.8 + s_1 &= -1.3 & \Rightarrow s_1 &= -0.5 \\ -0.8s_1 + 0.2r_0 &= 0.5 & \Rightarrow r_0 &= 0.5 \end{aligned}$$

Question

Consider the following discrete-time second-order plant model :

$$G(q^{-1}) = \frac{0.2q^{-1} + 0.3q^{-2}}{1 - 1.3q^{-1} + 0.4q^{-2}}$$

Design an RST controller with integrator and the same dynamics for tracking and regulation based on IMC technique ($P_f = 1$).

(A)
$R(q^{-1}) = 5 - 6.5q^{-1} + 2q^{-2}$
$S(q^{-1}) = 1 - q^{-1} - 1.5q^{-2}$
$T(q^{-1}) = 0.5$

(B)
$R(q^{-1}) = 0.2 - 0.26q^{-1} + 0.08q^{-2}$
$S(q^{-1}) = 1 - 0.04q^{-1} - 0.06q^{-2}$
$T(q^{-1}) = 0.02$

(C)
$R(q^{-1}) = 2(1 - 1.3q^{-1} + 0.4q^{-2})$
$S(q^{-1}) = 1 - 0.4q^{-1} - 0.6q^{-2}$
$T(q^{-1}) = 0.5$

(D)
$R(q^{-1}) = 2 - 2.6q^{-1} + 0.8q^{-2}$
$S(q^{-1}) = 1 - 0.4q^{-1} - 0.6q^{-2}$
$T(q^{-1}) = 0.2$

Internal Model Control (IMC)

Question

Solution :

- Solving $AS + BR = A$ gives $R = AR'$. Taking $R' = r_0'$ we have : $S(q^{-1}) = 1 - (0.2q^{-1} + 0.3q^{-2})r_0'$
- Integral action leads to $S(1) = 0$, therefore : $1 - (0.2 + 0.3)r_0' = 0$

$$\Rightarrow \quad r_0' = 2 \qquad \Rightarrow \quad R(q^{-1}) = 2(1 - 1.3q^{-1} + 0.4q^{-2})$$

- The $S(q^{-1})$ polynomial is computed as :

$$S(q^{-1}) = 1 - B(q^{-1})r_0' = 1 - 0.4q^{-1} - 0.6q^{-2}$$

- The $T(q^{-1})$ polynomial is computed as :

$$T(q^{-1}) = \frac{A(1)}{B(1)} = \frac{0.1}{0.5} = 0.2 \qquad \text{or} \qquad T(q^{-1}) = R(1) = 0.2$$