Digital Control

Control Systems

Fall 2024

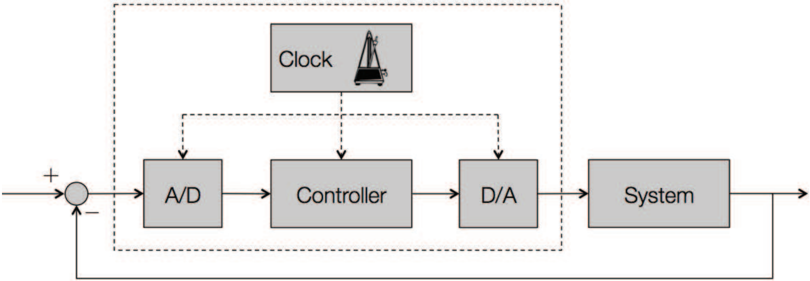
Outline

- Principles of digital control
  - Introduction
  - Sampling and reconstruction
  - Discrete-time systems
  - z-transform and its inverse
  - Discretization methods

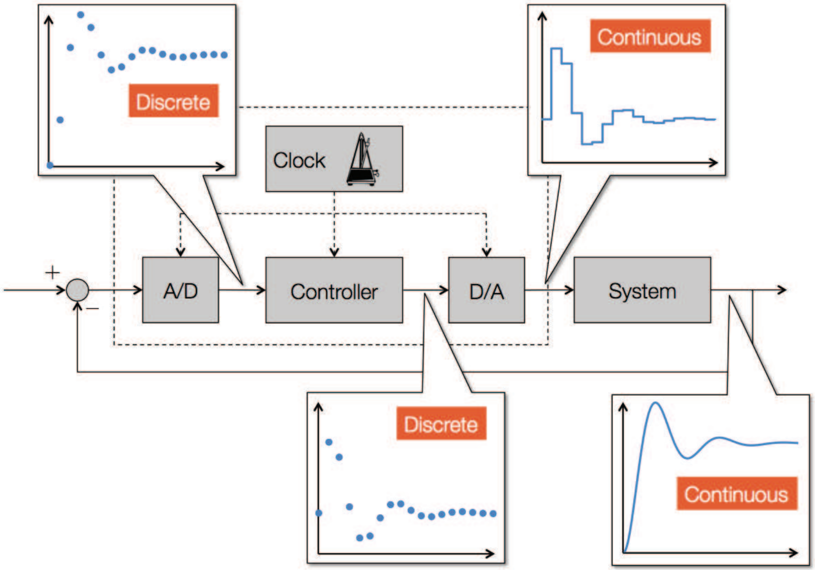
- Digital controller design
  - RST Controller
  - Pole Placement Technique

Introduction

- The first PI controllers were applied using pneumatic devices.
- A PID controller can be implemented using analogue electronic circuits (RLC+OpAmp).
- Now, microprocessors are used to implement the controllers, because
  - they are more flexible and cheaper,
  - they can control many subsystems,
  - more complex control algorithms can be applied,
  - new sensors provide numerical values.

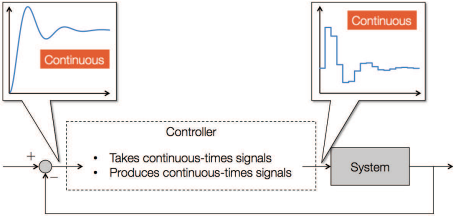


Introduction

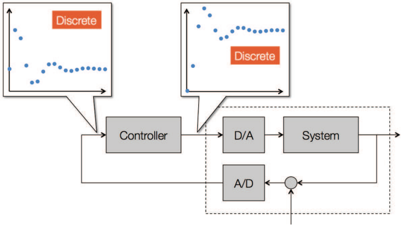


Digital versus Continuous-time Control

Perspective of the system : System sees the digital controller as a continuous-time device.



Perspective of the controller : Controller sees the system as a discrete-time entity.



Digital versus Continuous-time Control

Continuous-time control :

- Find a physical model from differential equations, or
- Find a simplified model from step or frequency response ;
- Design a continuous-time controller ;
- Apply the discretized controller.

Designed specifications are achieved only if the sampling time is very small

Digital Control :

- Find a discrete-time model by discretizing the physical model, or
- Identify directly from data a discrete-time model (System Identification Course) ;
- Design a discrete-time controller and apply it.

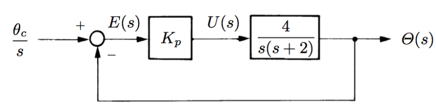
Designed specifications are achieved at sampling instants

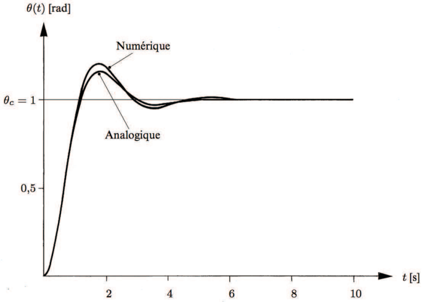
Digital versus Continuous-time Control

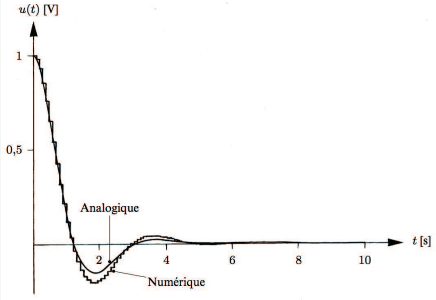
Example

Consider the position control of a DC motor with a proportional controller $K_p = 1$ (Clock pulse $h = 0.1$ s) :

$$e(t) = \theta_c - \theta(t) \qquad \text{and} \qquad u(t) = e(t)$$





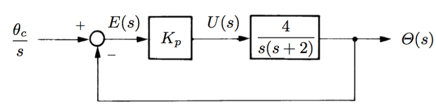


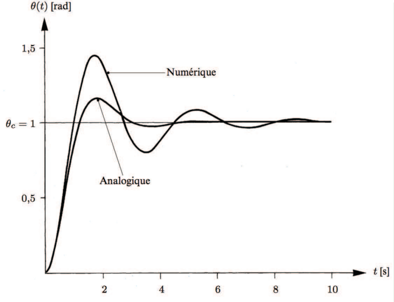
Digital versus Continuous-time Control

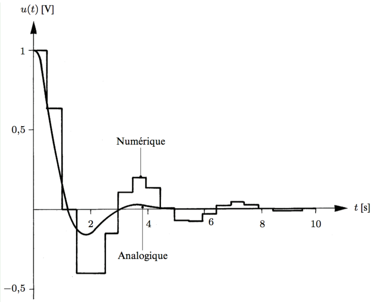
Example

Consider the position control of a DC motor with a proportional controller $K_p = 1$ (Clock pulse $h = 0.5$ s) :

$$e(t) = \theta_c - \theta(t) \qquad \text{and} \qquad u(t) = e(t)$$

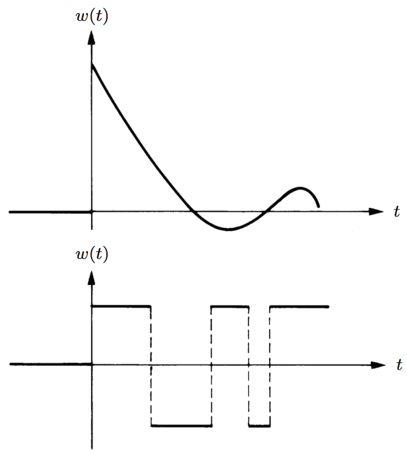






Continuous-time signal

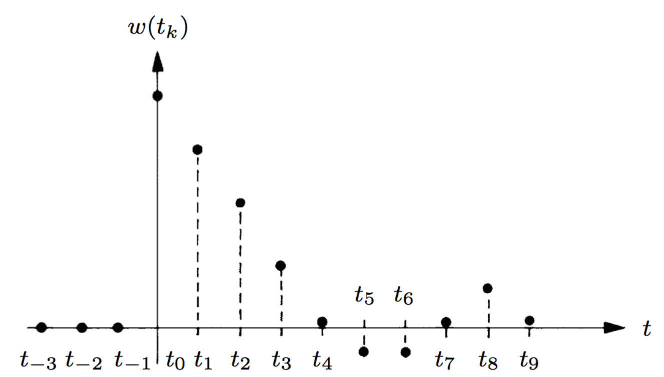
Continuous-time signal is a function defined as $w(t) : \mathbb{R} \to \mathbb{R}$.



Discrete-time signal

Discrete-time signal is defined as

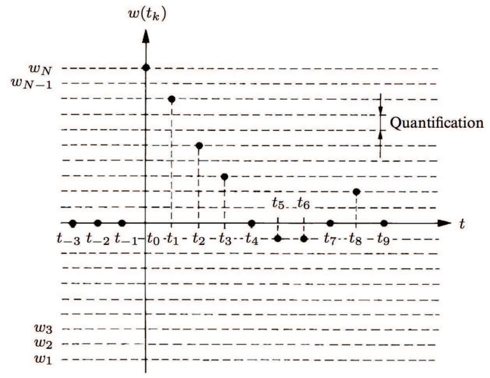
$$w(t_k) = w(kh) : \{k \in \mathbb{Z}\} \to \mathbb{R}$$



Digital signal

Digital signal is defined as

$$w(t_k) = w(kh) : \{k \in \mathbb{Z}\} \to \{w_1, w_2, \dots, w_N\} \subset \mathbb{R}$$



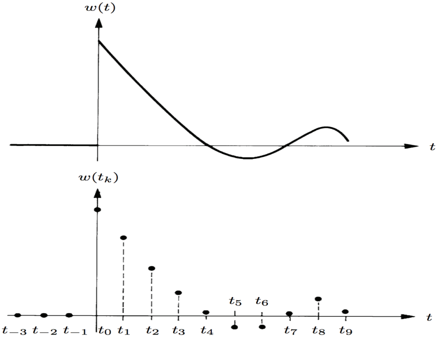
Sampling and Reconstruction

Sampling of a continuous-time signal : Normally we sample with a constant sampling period (période d'échantillonnage) h, i.e.

$$h = t_k - t_{k-1}, k \in \mathbb{Z}$$

Sampling frequency : (fréquence d'échantillonnage) $f_e = \dfrac{1}{h}$ Hz

or (pulsation d'échantillonnage) $\quad \omega_e = 2\pi f_e$ rad/s



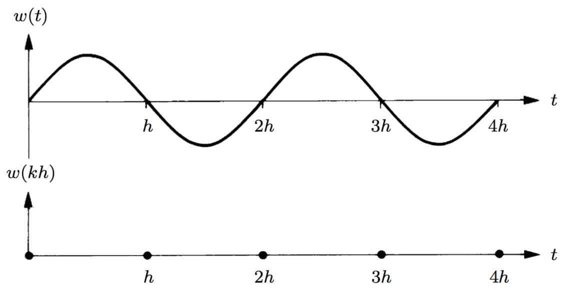
Sampling Theorem

Question

Can we capture all of the information in a continuous-time signal by sampling ?

It should depend on the sampling frequency and the frequency contents of the continuous-time signal.

Example



Sampling Theorem

Theorem (Shannon Sampling Theorem)

An analogue signal $w(t)$, whose Fourier transform is zero outside the interval $[-\omega_0\,,\,\omega_0]$, is completely defined by its sampled value $\{w(kh)\}$ if the sampling frequency satisfies

$$\omega_e > 2\omega_0$$

The signal $w(t)$ can be reconstructed by :

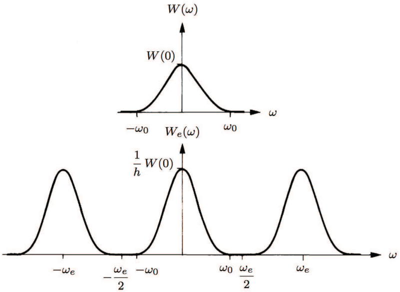
$$w(t) = \sum_{k=-\infty}^{\infty} w(kh) sinc\left(\frac{\omega_e(t-kh)}{2}\right)$$

Remark : This theorem is one of the most important results in the fields of information theory, communication and engineering.

Sampling Theorem

Proof : Let $W(\omega)$ be the Fourier transform of $w(t)$ and $W_e(\omega)$ a periodic function with period $\omega_e = \frac{2\pi}{h}$ defined as :

$$W_e(\omega) = \frac{1}{h}\sum_{i=-\infty}^{\infty} W(\omega + i\omega_e) \qquad \text{and} \qquad w(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W(\omega)e^{j\omega t}d\omega$$



This periodic function can be represented by a Fourier series :

$$W_e(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{-jkh\omega} \qquad \text{and} \qquad c_k = \frac{1}{2\pi}\int_{-\frac{\omega_e}{2}}^{\frac{\omega_e}{2}} hW_e(\omega)e^{jkh\omega}d\omega$$

Sampling Theorem

Proof (suit) : Given the fact that

$$W(\omega) = \left\{ \begin{array}{lll} hW_e(\omega) & \text{for} & |\omega| \leq \frac{\omega_e}{2} \\ 0 & \text{for} & |\omega| > \frac{\omega_e}{2} \end{array} \right. \qquad \Rightarrow c_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} W(\omega)e^{jkh\omega}d\omega$$

But, $c_k = w(kh)$, therefore :

$$W_e(\omega) = \sum_{k=-\infty}^{\infty} w(kh)e^{-jkh\omega}$$

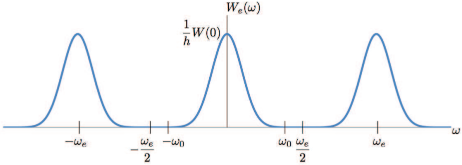
$W_e(\omega)$ can be computed only with information on sampled signal $w(kh)$

Now, we compute (recover) the signal $w(t)$:

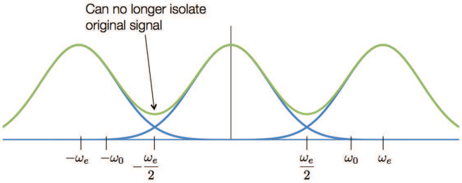
$$\begin{aligned} w(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} W(\omega)e^{j\omega t} = \frac{h}{2\pi}\int_{-\frac{\omega_e}{2}}^{\frac{\omega_e}{2}} W_e(\omega)e^{j\omega t}d\omega \\ &= \frac{h}{2\pi}\int_{-\frac{\omega_e}{2}}^{\frac{\omega_e}{2}} \sum_{k=-\infty}^{\infty} w(kh)e^{-jkh\omega}e^{j\omega t}d\omega = \sum_{k=-\infty}^{\infty} w(kh)\frac{h}{2\pi}\int_{-\frac{\omega_e}{2}}^{\frac{\omega_e}{2}} e^{j(t-kh)\omega}d\omega \\ &= \sum_{k=-\infty}^{\infty} w(kh)\frac{h}{2\pi}\int_{-\infty}^{\infty} rect(\omega_e)e^{j(t-kh)\omega}d\omega = \sum_{k=-\infty}^{\infty} w(kh)\text{sinc}\left(\frac{\omega_e(t-kh)}{2}\right) \end{aligned}$$

Anti-aliasing Filter

What happens if the sampling frequency is not large enough ?



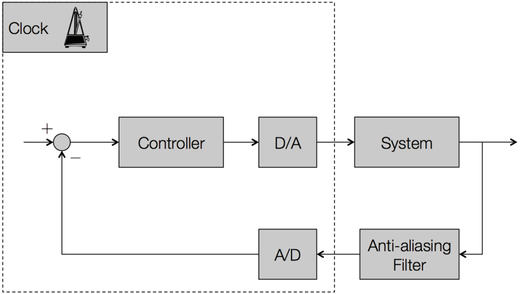
$\omega_e > 2\omega_0$

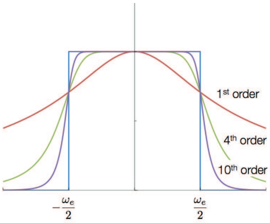


$\omega_e < 2\omega_0$

Apply a lowpass analog filter to cut all frequencies greater than $\omega_e/2$.
This filter is called the anti-aliasing filter.

Anti-aliasing Filter





- Real signals are not band limited so there is always an aliasing effect.
- Anti-aliasing filter should be applied **before** sampling.
- A perfect lowpass filter does not exist.
- Higher order filters give better attenuation but introduce more phase lag or delay in the signal (there is always a trade-off).

Reconstruction

Reconstruction is to recover the analogue signal $w(t)$ from its samples $\{w(kh)\}$.

Shannon reconstruction : is based on the Shannon Theorem.

$$w(t) = \sum_{k=-\infty}^{\infty} w(kh)\text{sinc}\left(\frac{\omega_e(t-kh)}{2}\right)$$

- It is too complicated.
- It can only be applied in off-line applications in signal processing.
- It cannot be applied in real-time because the computation of $w(t)$ depends the future samples $w(kh), kh > t$.
- In control applications an approximation of $w(t)$ is constructed by a **hold** function.

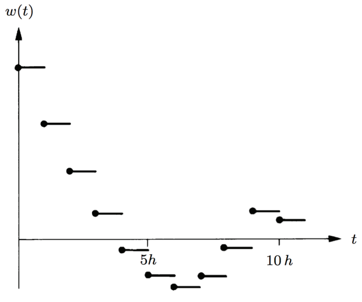
Reconstruction

Zero-Order-Hold (ZOH)

The signal remains constant between two samples.

$$w(t) = w(kh) \qquad t \in [kh\,,\ kh + h[$$

Example (ZOH)



Remark : Fast variations at the sampling instants add high frequency contents to the signal but they are usually filtered by the process dynamics, which is a lowpass filter.

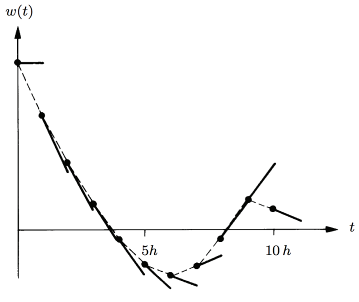
Reconstruction

First-Order-Hold

The signal is approximated by a linear extrapolation.

$$w(t) = w(kh) + \frac{t-kh}{h}\left(w(kh) - w(kh-h)\right) \qquad t \in [kh\,,\; kh+h[$$

Example (FOH)



Remark : First-order-hold reduces the jumps in the signal but will amplify the noise effect because of its derivative action.

Choice of Sampling Period

According to Shannon Theorem : The sampling frequency should be greater than $2\omega_0$. However,

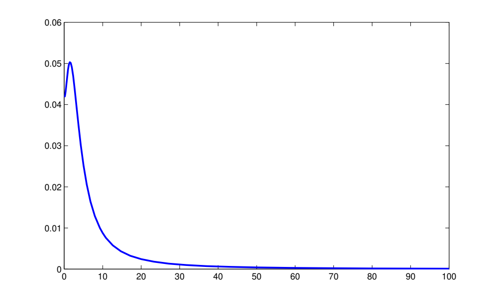
- The physical signals are not band limited ($\omega_0 = \infty$).
- The Shannon reconstruction cannot be used in real-time applications.
- In control applications, the sampling frequency is chosen according to the frequency response of the desired closed-loop system (a low pass filter). It is usually chosen 20 to 50 times of the desired closed-loop bandwidth.
- The sampling period can be chosen from the step response. The rule of thumb is to have between 5 to 10 samples during the rise-time.
- The sampling frequency should not be too large to avoid the numerical problems and hardware costs.

Sampling Period

Find a sampling period for the control of following system :

$$G(s) = \frac{s+1}{(s+2)(s+3)(s+4)}$$

1. Frequency response in a linear scale : $\omega_e > 2\omega_0$



(A) : $\omega_e \approx 20$ rad/s

(B) : $\omega_e \approx 100$ rad/s

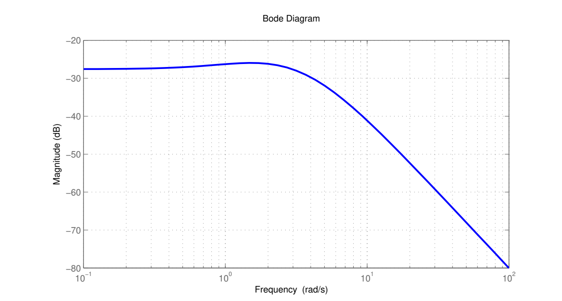
(C) : $\omega_e \approx 200$ rad/s

(D) : $\omega_e \approx 1000$ rad/s

Choice of Sampling Period

Sampling Period

2. Bode diagram : $\omega_e = (20 \textbf{ to } 50)\times$ **Bandwidth**

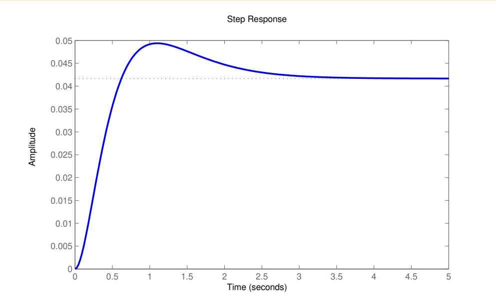


(A) : $\omega_e \approx (10 \text{ to } 50)$ rad/s
(B) : $\omega_e \approx (20 \text{ to } 100)$ rad/s
(C) : $\omega_e \approx (80 \text{ to } 200)$ rad/s
(D) : $\omega_e \approx (120 \text{ to } 300)$ rad/s

Choice of Sampling Period

Sampling Period

3. Step Response : (5 to 10) samples during the rise time.



(A) : $h \approx (0.05 \text{ to } 0.1)$ s

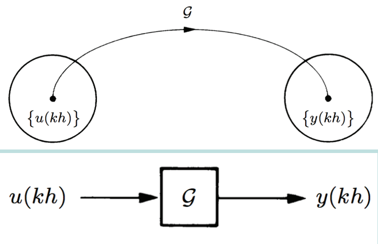
(B) : $h \approx (0.01 \text{ to } 0.02)$ s

(C) : $h \approx (0.2 \text{ to } 0.4)$ s

(D) : $h \approx (0.5 \text{ to } 1)$ s

Discrete-time systems

A discrete system maps a discrete signal to a discrete signal.



Dynamic system

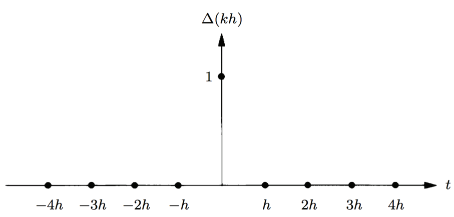
The output at instant $k_0 h$ depends on the present and past inputs $\{u(kh) : k \leq k_0\}$.

Discrete-Time Systems

Unit impulse

Unit impulse or Kronecker delta is a discrete signal defined as :

$$\Delta(kh) = \begin{cases} 1 & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k \neq 0 \end{cases}$$



Impulse Response

The response of a discrete system $\mathcal{G}$ to a unit impulse is called the impulse response and is given by $\{g(kh)\}$.

Discrete-Time Systems

Linear system

A linear system follows the superposition principle.

$$\mathcal{G}\left(\{u_1(kh)\} + \{u_2(kh)\}\right) = \mathcal{G}\left(\{u_1(kh)\}\right) + \mathcal{G}\left(\{u_2(kh)\}\right)$$

$$\mathcal{G}\left(a\{u(kh)\}\right) = a\mathcal{G}\left(\{u(kh)\}\right)$$

Time-invariant System

A discrete system $\mathcal{G}$ is stationary or time-invariant if its response to a shifted unit impulse by dh is shifted by dh :

$$\{\Delta(kh)\} \to \{g(kh)\} \quad \Rightarrow \quad \{\Delta(kh+dh)\} \to \{g(kh+dh)\} \quad \forall d \in \mathbb{Z}$$

Discrete-Time Systems

Theorem (Convolution Product)

The response of a Linear Time-Invariant (LTI) system to any input is given by the following convolution product :

$$y(kh) = u(kh) * g(kh) = \sum_{\ell=0}^{k} u(\ell h) g(kh - \ell h)$$

Proof : Any input signal can be written as the shifted sum of weighted unit impulses $\{u(kh)\} = \sum_{\ell=0}^{\infty} u(\ell h)\{\Delta(kh - \ell h)\}$. Using the properties of LTI systems we have

$$\begin{aligned} \{y(kh)\} &= \mathcal{G}\{u(kh)\} = \mathcal{G}\left(\sum_{\ell=0}^{\infty} u(\ell h)\{\Delta(kh-\ell h)\}\right) \\ &= \sum_{\ell=0}^{\infty} u(\ell h)\mathcal{G}\{\Delta(kh-\ell h)\} = \sum_{\ell=0}^{\infty} u(\ell h)\{g(kh-\ell h)\} \end{aligned}$$

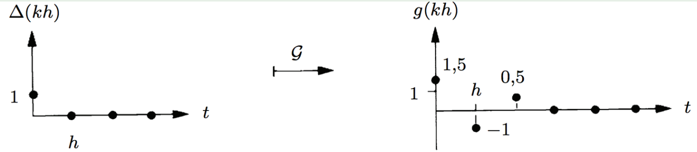
Because of the system's causality, the output at instant kh can be written as :

$$y(kh) = \sum_{\ell=0}^{k} u(\ell h) g(kh - \ell h)$$

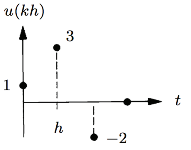
Discrete-Time Systems

Example

Given the impulse response of a discrete system as



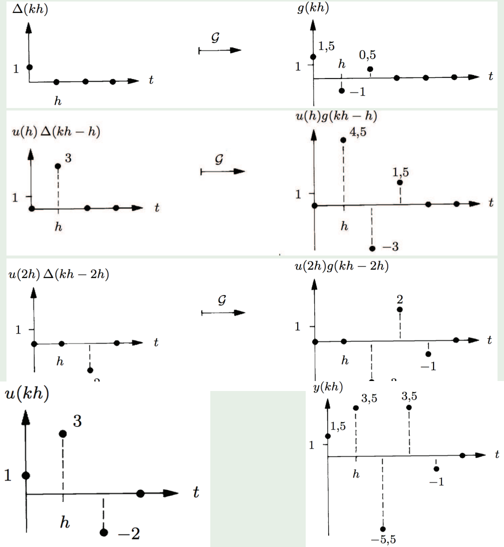
Compute the response of the system to the following input.



The input signal at instant kh can be written as :

$$u(kh) = \Delta(kh) + 3\Delta(kh - h) - 2\Delta(kh - 2h)$$

Example



Δ(kh)
1
h
t
𝒢
g(kh)
1,5
1
h
0,5
−1
t
u(h) Δ(kh − h)
3
1
h
t
𝒢
u(h)g(kh − h)
4,5
1,5
1
h
−3
t
u(2h) Δ(kh − 2h)
1
h
t
𝒢
u(2h)g(kh − 2h)
2
1
h
−1
t
u(kh)
3
1
h
−2
t
y(kh)
3,5
3,5
1,5
1
h
−1
−5,5
t

Discrete-Time Systems

Usually the simplified notation $u(k)$ is used instead of $u(kh)$

Properties of convolution

Commutativity :

$$u(k) * v(k) = v(k) * u(k)$$

Distributivity :

$$u(k) * (v(k) + w(k)) = u(k) * v(k) + u(k) * w(k)$$

Associativity :

$$u(k) * (v(k) * w(k)) = (u(k) * v(k)) * w(k)$$

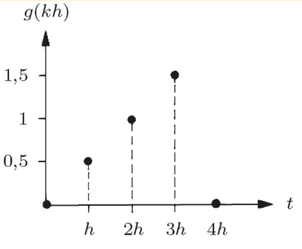
Identity : The identity element is the unit impulse.

$$u(k) * \Delta(k) = u(k)$$

Discrete-Time Systems

Question

The impulse response $\{g(kh)\}$ of an LTI system is given. Compute the response of the system when excited by the signal $\{\mathbf{1}, 1, 1, 1, 0, 0, ...\}$.



(A) $\{y(kh)\} = \{\mathbf{0}, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, \ldots,\}$
(B) $\{y(kh)\} = \{\mathbf{0}, 0.5, 1.5, 2.5, 3, 2.5, 1.5, 0, 0, \ldots,\}$
(C) $\{y(kh)\} = \{\mathbf{0}, 0.5, 1.5, 3, 3, 2.5, 1.5, 0, 0, \ldots,\}$
(D) $\{y(kh)\} = \{\mathbf{0}, 0.5, 1, 1.5, 0.5, 1, 1.5, 0.5, 1, \ldots,\}$

Discrete-Time Systems

Linear Difference Equation

A linear difference equation with constant coefficients of order n represents the relation between the successive values of input and output of a system :

$$
\begin{aligned}
y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) \\
= b_0 u(k) + b_1 u(k-1) + \cdots + b_m u(k-m)
\end{aligned}
$$

or equivalently by :

$$y(k) = -\sum_{i=1}^{n} a_i y(k-i) + \sum_{j=0}^{m} b_j u(k-j)$$

- If all of the initial conditions $y(-1), y(-2), \ldots, y(-n)$ and the values of $u(-1), u(-2), \ldots, u(-n)$ are zeros, the output at each instant can be computed recursively based only on the past inputs.
- Systems defined by linear difference equations are LTI systems.

Delay operator

The delay operator q^{-n} is a function that maps the discrete signal $\{w(k)\}$ to $\{w(k-n)\}$:

$$q^{-n}\{w(k)\} = \{w(k-n)\}$$

Therefore a difference equation can be represented as :

$$(1 + a_1 q^{-1} + a_2 q^{-2} + \cdots + a_n q^{-n})y(k) = (b_0 + b_1 q^{-1} + \cdots + b_m q^{-m})u(k)$$

Example

Show the difference equation of a PD controller by the delay operator :

$$u(k) = K_p \left(e(k) + T_d \frac{e(k) - e(k-1)}{h} \right)$$

$$u(k) = \left(K_p + \frac{K_p T_d (1 - q^{-1})}{h} \right) e(k)$$

The z-Transform

Continuous-time and discrete-time systems are very similar

Continuous time System	Discrete-time System
$g(t)$: Response to a Dirac impulse	$g(k)$: Response to a unit impulse
Convolution integral $y(t) = \int_0^t u(\tau)g(t-\tau)d\tau$	Convolution sum $y(k) = \sum_{\ell=0}^{k} u(\ell)g(k-\ell)$
Differential equation $\frac{dy(t)}{d(t)} + ay(t) = bu(t)$	Difference equation $y(k) + cy(k-1) = du(k)$
Laplace Transform $Y(s) = \mathcal{L}(y(t))$ Transfer function $G(s) = \mathcal{L}(g(t))$ $Y(s) = G(s)U(s)$	?

The z-Transform

z-Transform

The z transform of a discrete signal $\{w(kh)\}$ is denoted by $W(z)$ or $\mathcal{Z}\{w(kh)\}$ and is defined by the following sum, where z is a complex variable $z \in \mathbb{C}$:

$$W(z) = \mathcal{Z}\{w(kh)\} = \sum_{k=0}^{\infty} w(kh) z^{-k}$$

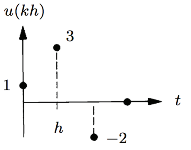
Remarks :

- This is a unilateral transformation (from $k = 0$). In control theory, it is supposed that $w(kh) = 0$, $\forall k < 0$.
- The region of convergence is a set in the complex plane for which the z-transform summation converges.
- There exists a real number $r \geq 0$ such that the above sum converges for all $|z| > r$ and diverges for all $|z| < r$. The number r is called the convergence radius.

The z-Transform

Example

Compute the z-transform of the following signal :



$$\mathcal{Z}\{u(kh)\} = 1 + 3z^{-1} - 2z^{-2} \qquad |z| > 0 = r$$

Example

Compute the z-transform of the following signal :

$$w(k) = a^k \qquad a \in \mathbb{C}, \quad a \neq 0 \quad k \geq 0$$

$$\mathcal{Z}\{a^k\} = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} \left(az^{-1}\right)^k = \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \qquad |z| > |a| = r$$

Properties of the z-Transform

Theorem (Linearity)

The z-transform is a linear operator, i.e.

$$\mathcal{Z}\{w_1(k) + w_2(k)\} = \mathcal{Z}\{w_1(k)\} + \mathcal{Z}\{w_2(k)\}$$

$$\mathcal{Z}\{aw_1(k)\} = a\mathcal{Z}\{w_1(k)\}$$

The ROC of $\mathcal{Z}\{w_1(k) + w_2(k)\}$ is the intersection of the ROC $\mathcal{Z}\{w_1(k)\}$ and $\mathcal{Z}\{w_2(k)\}$.

Proof :

$$\begin{aligned}\mathcal{Z}\{w_1(k) + w_2(k)\} &= \sum_{k=0}^{\infty}(w_1(k) + w_2(k))z^{-k} \\ &= \sum_{k=0}^{\infty} w_1(k)z^{-k} + \sum_{k=0}^{\infty} w_2(k)z^{-k} = \mathcal{Z}\{w_1(k)\} + \mathcal{Z}\{w_2(k)\}\end{aligned}$$

$$\mathcal{Z}\{aw_1(k)\} = \sum_{k=0}^{\infty} aw_1(k)z^{-k} = a\sum_{k=0}^{\infty} w_1(k)z^{-k} = a\mathcal{Z}\{w_1(k)\}$$

Properties of the z-Transform

Example

Compute the z-transform of $w(kh) = \sin(\omega kh)$.
Because of the linearity of the z-transform, we have :

$$
\begin{aligned}
\mathcal{Z}\{\sin(\omega kh)\} &= \mathcal{Z}\left\{\frac{e^{j\omega kh} - e^{-j\omega kh}}{2j}\right\} \\
&= \frac{1}{2j}\left(\mathcal{Z}\{e^{j\omega kh}\} - \mathcal{Z}\{e^{-j\omega kh}\}\right)
\end{aligned}
$$

Taking $a = e^{j\omega h}$ then $a = e^{-j\omega h}$ and using the result of the previous example, we have

$$
\mathcal{Z}\{\sin(\omega kh)\} = \frac{1}{2j}\left(\frac{z}{z - e^{j\omega h}} - \frac{z}{z - e^{-j\omega h}}\right) = \frac{\sin(\omega h)z}{z^2 - 2\cos(\omega h)z + 1}
$$

Properties of the z-Transform

Question

Compute the z-transform of $w(kh) = \cos(\omega kh)$.

(A) $\dfrac{\cos(\omega h)z}{z^2 - 2\sin(\omega h)z + 1}$ **(B)** $\dfrac{2z^2 - 2\cos(\omega h)z}{z^2 - 2\cos(\omega h)z + 1}$

(C) $\dfrac{z^2 - \cos(\omega h)z}{z^2 - 2\cos(\omega h)z + 1}$ **(D)** $\dfrac{z^2 + \cos(\omega h)z}{z^2 - 2\sin(\omega h)z + 1}$

Properties of the z-Transform

Theorem (Time delay)

The z-transform of a delayed discrete signal is given by :

$$\mathcal{Z}\{w(k-d)\} = z^{-d}\mathcal{Z}\{w(k)\} = z^{-d}W(z) \qquad \textit{where} \qquad d \in \mathbb{N}$$

Proof :

$$\mathcal{Z}\{w(k-d)\} = \sum_{k=0}^{\infty} w(k-d)z^{-k} = z^{-d}\sum_{k=d}^{\infty} w(k-d)z^{-(k-d)}$$

$$= z^{-d}\sum_{k'=0}^{\infty} w(k')z^{-k'} = z^{-d}W(z)$$

Example

$$\mathcal{Z}\{\Delta(k-1)\} = z^{-1}$$

Note that $\mathcal{Z}\{\Delta(k-1)\}$ is not defined for $z=0$, while $\mathcal{Z}\{\Delta(k)\}$ it is.

Properties of the z-Transform

Theorem (Complex Derivative)

The z-transform has the following property :

$$\mathcal{Z}\{kw(k)\} = -z\frac{dW(z)}{dz}$$

Proof :

$$\begin{aligned} -z\frac{dW(z)}{dz} &= -z\frac{d}{dz}\sum_{k=0}^{\infty} w(k)z^{-k} = z\sum_{k=0}^{\infty} kw(k)z^{-k-1} \\ &= \sum_{k=0}^{\infty} kw(k)z^{-k} = \mathcal{Z}\{kw(k)\} \end{aligned}$$

Example

Knowing that the z transform of the unit step, $w(k) = 1, k \geq 0$ is given by $W(z) = \dfrac{z}{z-1}$, find the z-transform of a ramp signal $r(k) = k$.

$$\mathcal{Z}\{r(k)\} = \mathcal{Z}\{kw(k)\} = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2}$$

Properties of the z-Transform

Theorem (Initial Value)

The initial value $w(0)$ of a discrete signal $\{w(k)\}$ can be evaluated from $W(z)$ using :

$$w(0) = \lim_{z \to \infty} W(z)$$

Proof :

$$\begin{aligned} \lim_{z \to \infty} W(z) &= \lim_{z \to \infty} \lim_{n \to \infty} \sum_{k=0}^{n} w(k) z^{-k} \\ &= \lim_{n \to \infty} \lim_{z \to \infty} \sum_{k=0}^{n} w(k) z^{-k} = \lim_{n \to \infty} w(0) = w(0) \end{aligned}$$

Example

Compute the initial value of $w(k)$, where

$$W(z) = \frac{b_0 z + b_1}{z + a_1}$$

$$w(0) = \lim_{z \to \infty} W(z) = b_0$$

Properties of the z-Transform

Consider the following rational function :

$$W(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

Properness

$W(z)$ is called **proper** if $n \geq m$, **strictly proper** if $n > m$ and **biproper** if $n = m$.

zero/pole

The roots of the denominators are called the **poles** and the roots of the numerator the **zeros** of $W(z)$.

Monic Polynomials

The polynomial in the denominator is called **monic** because the coefficient of z^n is equal to 1. The rational functions, without loss of generality are usually represented by a monic denominator.

Properties of the z-Transform

Theorem (Final Value)

If $W(z)$ is a proper rational function with all poles inside the unit circle, except possibly one simple pole at 1, then

$$\lim_{k\to\infty} w(k) = \lim_{z\to 1}(z-1)W(z)$$

Proof :

$$\begin{aligned}
\lim_{z\to 1}(z-1)W(z) &= \lim_{z\to 1}\frac{z-1}{z}W(z) = \lim_{z\to 1}(1-z^{-1})W(z) \\
&= \lim_{n\to\infty}\lim_{z\to 1}\sum_{k=0}^{n}(w(k)-w(k-1))z^{-k} \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}(w(k)-w(k-1)) \\
&= \lim_{n\to\infty}(w(0)-0+w(1)-w(0)+\cdots+w(n)-w(n-1)) \\
&= \lim_{n\to\infty} w(n)
\end{aligned}$$

Properties of the z-Transform

Final Value

Find the final value of $w(k)$ if $W(z) = \dfrac{0.5z}{(z-1)(z-0.5)}$.

(A) $w(\infty) = -1$ **(B)** $w(\infty) = 0$

(C) $w(\infty) = \infty$ **(D)** $w(\infty) = 1$

Properties of the z-Transform

Final Value

Find the final value of $w(k)$ if $W(z) = \dfrac{z}{z+2}$.

(A)	$w(\infty) = 1/3$	**(B)**	$w(\infty) = 0$
(C)	$w(\infty) = \infty$	**(D)**	None of the above

Properties of the z-Transform

Theorem (Convolution)

$$y(k) = \sum_{\ell=0}^{k} u(\ell)g(k-\ell) \quad \Rightarrow \quad Y(z) = G(z)U(z)$$

Proof :

$$\begin{aligned}
G(z)U(z) &= \left(\sum_{k=0}^{\infty} g(k)z^{-k}\right)\left(\sum_{k=0}^{\infty} u(k)z^{-k}\right) \\
&= (g(0)+g(1)z^{-1}+\cdots)(u(0)+u(1)z^{-1}+\cdots) \\
&= u(0)g(0) + (u(0)g(1)+u(1)g(0))z^{-1} \\
&\quad +(u(0)g(2)+u(1)g(1)+u(2)g(0))z^{-2}+\cdots \\
&\quad + \left(\sum_{\ell=0}^{k} u(\ell)g(k-\ell)\right) z^{-k} + \cdots \\
&= \sum_{k=0}^{\infty}\left(\sum_{\ell=0}^{k} u(\ell)g(k-\ell)\right) z^{-k} = \sum_{k=0}^{\infty} y(k)z^{-k} = Y(z)
\end{aligned}$$

Properties of the z-Transform

Example

Compute the z-transform of the accumulative sum of a signal.

Solution : Take $u(k)$ as a discrete unit step with $U(z) = \dfrac{z}{z-1}$. Then

$$\mathcal{Z}\left\{\sum_{\ell=0}^{k} w(\ell)\right\} = \mathcal{Z}\left\{\sum_{\ell=0}^{k} u(\ell)w(k-\ell)\right\} = U(z)W(z) = \frac{z}{z-1}W(z)$$

A pole at 1 represents the integrator effect.

Example

Compute the z-transform of the difference between two consecutive samples of a discrete signal.

Solution : $\mathcal{Z}\{w(k) - w(k-1)\} = W(z) - z^{-1}W(z) = \dfrac{z-1}{z}W(z)$

A zero at 1 represents the derivative effect.

Table of z-Transforms

Tableau 4.3 Dictionnaire des transformées de Laplace et en z.

$w(t)$	$\mathcal{L}(w(t))$	$w(kh)$	$\mathcal{Z}\{w(kh)\}$
$\delta(t)$	1		
		$\Delta(kh)$	1
1	$\frac{1}{s}$	1	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	kh	$\frac{hz}{(z-1)^2}$
$\frac{1}{2}t^2$	$\frac{1}{s^3}$	$\frac{1}{2}(kh)^2$	$\frac{h^2 z(z+1)}{2(z-1)^3}$
e^{-at}	$\frac{1}{s+a}$	e^{-akh}	$\frac{z}{z-\mathrm{e}^{-ah}}$
$t\,\mathrm{e}^{-at}$	$\frac{1}{(s+a)^2}$	$kh\,\mathrm{e}^{-akh}$	$\frac{h\,\mathrm{e}^{-ah}z}{\left(z-\mathrm{e}^{-ah}\right)^2}$

Table of z-Transforms

Tableau 4.3 Dictionnaire des transformées de Laplace et en z.

$w(t)$	$\mathcal{L}(w(t))$	$w(kh)$	$\mathcal{Z}\{w(kh)\}$
$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$	$\sin(\omega kh)$	$\dfrac{\sin(\omega h)z}{z^2-2\cos(\omega h)z+1}$
$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$	$\cos(\omega kh)$	$\dfrac{z(z-\cos(\omega h))}{z^2-2\cos(\omega h)z+1}$
$\mathrm{e}^{-at}\sin(\omega t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$	$\mathrm{e}^{-akh}\sin(\omega kh)$	$\dfrac{\mathrm{e}^{-ah}\sin(\omega h)z}{z^2-2\mathrm{e}^{-ah}\cos(\omega h)z+\mathrm{e}^{-2ah}}$
$\mathrm{e}^{-at}\cos(\omega t)$	$\dfrac{s+a}{(s+a)^2+\omega^2}$	$\mathrm{e}^{-akh}\cos(\omega kh)$	$\dfrac{z\left(z-\mathrm{e}^{-ah}\cos(\omega h)\right)}{z^2-2\mathrm{e}^{-ah}\cos(\omega h)z+\mathrm{e}^{-2ah}}$
		a^k	$\dfrac{z}{z-a}$
		ka^{k-1}	$\dfrac{z}{(z-a)^2}$
		$\dfrac{1}{2}k(k-1)a^{k-2}$	$\dfrac{z}{(z-a)^3}$

Inverse z-transform

The inverse z-transform of a function $W : \mathbb{C} \to \mathbb{C}$ is a discrete signal $w(k)$, denoted by $\mathcal{Z}^{-1}(W(z))$ such that $\mathcal{Z}\{w(k)\} = W(z)$.

Example

Compute the inverse z-transform of $W(z) = \dfrac{3}{(z+1)(z-2)}$

Solution : We use a similar method to the inverse of Laplace transform.

$$\frac{W(z)}{z} = \frac{3}{z(z+1)(z-2)} = -\frac{1.5}{z} + \frac{1}{z+1} + \frac{0.5}{z-2}$$

$$\Rightarrow \quad W(z) = -1.5 + \frac{z}{z+1} + \frac{0.5z}{z-2}$$

$$\Rightarrow \quad w(k) = -1.5\Delta(k) + (-1)^k + 0.5(2)^k \quad k \geq 0$$

Inverse z-Transform

Computing the inverse z-transform of a rational function :

$$W(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

1. Compute the poles p_1 to p_n, where $p_i \in \mathbb{C}$.
2. Re-write $W(z)$ in the following form :

$$W(z) = c_0 + \frac{c_1 z}{z - p_1} + \frac{c_2 z}{z - p_2} + \cdots + \frac{c_n z}{z - p_n}$$

3. Take $c_0 = W(0)$ and compute the constants $c_i, i = 1, \ldots, n$ by :

$$c_i = \lim_{z \to p_i} \left(\frac{z - p_i}{z} W(z) \right)$$

4. Compute the inverse transform from the Table.

$$w(k) = c_0 \Delta(k) + c_1 p_1^k + c_2 p_2^k + \cdots + c_n p_n^k \qquad k \geq 0$$

Inverse z-Transform

Example

Compute the inverse z-transform of $W(z) = \dfrac{z+3}{z^2 - 3z + 2}$.

1. Compute the poles : $\quad z^2 - 3z + 2 = (z-1)(z-2)$.
2. Re-write $W(z) = c_0 + \dfrac{c_1 z}{z-1} + \dfrac{c_2 z}{z-2}$
3. Take $c_0 = W(0) = 1.5$ and compute c_1 and c_2 :

$$c_1 = \lim_{z \to 1} \left(\frac{z-1}{z} \frac{z+3}{(z-1)(z-2)} \right) = -4$$

$$c_2 = \lim_{z \to 2} \left(\frac{z-2}{z} \frac{z+3}{(z-1)(z-2)} \right) = 2.5$$

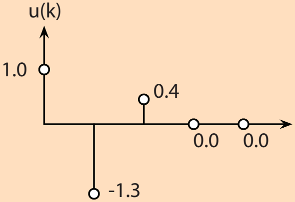
4. Compute $w(k)$ from the Table :

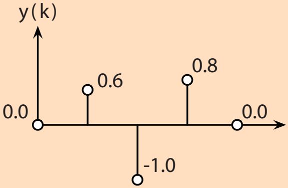
$$w(k) = 1.5\Delta(k) - 4 + 2.5(2)^k \qquad k \geq 0$$

Inverse z-Transform

Question : Exam 2015

Soit $u(k)$ et $y(k)$, respectivement, l'entrée et la sortie d'un systéme discret :





1. Trouver la fonction de transfert du systéme.

(A) $\dfrac{0.6z^2 - z + 0.8}{z(z^2 - 1.3z + 0.4)}$ **(B)** $\dfrac{0.6z^2 - z + 0.8}{z^2 - 1.3z + 0.4}$

(C) $\dfrac{z^2 - 1.3z + 0.4}{z(0.6z^2 - z + 0.8)}$ **(D)** None of the above

Inverse z-Transform

Question : Exam 2015

1. Calculer la réponse indicielle (*step response*) du systéme.

(A) $y(k) = 2 - 8(-0.8)^k + 6(-0.5)^k$

(B) $y(k) = 2 - 8(0.8)^k + 6(0.5)^k$

(C) $y(k) = -2\Delta(k) + 4 - 8(0.8)^k + 6(0.5)^k$

(D) None of the above

There are three special cases :

1. Complex poles

$$W(z) = \cdots + \frac{cz}{z-p} + \frac{c'z}{z-\bar{p}} + \cdots$$

   where $\bar{p}$ is the complex conjugates of p.

2. Multiple poles at zero

$$W(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots + \frac{c_\ell}{z^\ell} + \cdots$$

3. Multiple non-zero poles

$$W(z) = \cdots + \frac{c_1 z}{(z-p)} + \frac{c_2 z}{(z-p)^2} + \cdots + \frac{c_\ell z}{(z-p)^\ell} + \cdots$$

Inverse z-Transform

1. Complex poles : $W(z) = \cdots + \dfrac{cz}{z-p} + \dfrac{c'z}{z-\bar{p}} + \cdots$

We have

$$c = \lim_{z\to p}\left(\frac{z-p}{z}W(z)\right) \qquad c' = \lim_{z\to \bar{p}}\left(\frac{z-\bar{p}}{z}W(z)\right)$$

Since $W(z)$ has real coefficients, $c' = \bar{c}$. Then,

$$w(k) = \cdots + cp^k + \bar{c}\bar{p}^k + \cdots \qquad k \geq 0$$

If we take $p = re^{j\omega}$, $c = x + jy$ and $\phi = \arctan(y/x)$, we obtain :

$$\begin{aligned} w(k) &= \cdots + (x+jy)r^k e^{jk\omega} + (x-jy)r^k e^{-jk\omega} + \cdots \\ &= \cdots + r^k x(e^{jk\omega} + e^{-jk\omega}) + jr^k y(e^{jk\omega} - e^{-jk\omega}) + \cdots \\ &= \cdots + r^k(2x\cos(k\omega) - 2y\sin(k\omega)) + \cdots \\ &= \cdots + 2|c|r^k\cos(k\omega + \phi) + \cdots \qquad k \geq 0 \end{aligned}$$

Example (Complex poles)

Compute the inverse z-transform for $W(z) = \dfrac{3z^2 + 0.5z}{z^2 - z + 0.5}$

1. Compute the poles :
   $z^2 - z + 0.5 = (z - (0.5 + 0.5j))(z - (0.5 - 0.5j))$.
2. Re-write $W(z) = c_0 + \dfrac{cz}{z - (0.5 + 0.5j)} + \dfrac{\bar{c}z}{z - (0.5 - 0.5j)}$
3. Take $c_0 = W(0) = 0$ and

$$c = \lim_{z \to (0.5+0.5j)} \left(\frac{z - (0.5 + 0.5j)}{z} \frac{3z^2 + 0.5z}{z^2 - z + 0.5} \right) = 1.5 - 2j$$

4. Take $p = 0.5 + 0.5j = \dfrac{\sqrt{2}}{2} e^{j\frac{\pi}{4}}$.
5. Finally : $w(k) = \left(\dfrac{\sqrt{2}}{2}\right)^k \left(3\cos\left(k\dfrac{\pi}{4}\right) + 4\sin\left(k\dfrac{\pi}{4}\right)\right) \quad k \geq 0$

Inverse z-Transform

2. Multiple poles at zero : $W(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots + \frac{c_\ell}{z^\ell} + \cdots$

We have

$$z^\ell W(z) = c_0 z^\ell + c_1 z^{\ell-1} + c_2 z^{\ell-2} + \cdots + c_{\ell-1} z + c_\ell + \cdots$$

Therefore : $c_\ell = \lim_{z\to 0} z^\ell W(z)$. Let's take the derivative of $z^\ell W(z)$:

$$\frac{d}{dz} z^\ell W(z) = c_0 \ell z^{\ell-1} + c_1(\ell-1) z^{\ell-2} + \cdots + 2c_{\ell-2} z + c_{\ell-1} + \cdots$$

Then : $c_{\ell-1} = \lim\limits_{z\to 0} \frac{d}{dz} z^\ell W(z)$. For the general case :

$$c_{\ell-i} = \lim_{z\to 0} \left(\frac{1}{i!} \frac{d^i}{dz^i} \left(z^\ell W(z) \right) \right) \qquad i = 0, 1, \ldots, \ell$$

From the Table $\mathcal{Z}^{-1}\{z^{-i}\} = \Delta(k-i)$ and therefore

$$w(k) = \cdots + c_0 \Delta(k) + c_1 \Delta(k-1) + c_2 \Delta(k-2) + \cdots + c_\ell \Delta(k-\ell) + \cdots \quad k \geq 0$$

Inverse z-Transform

Example (Multiple poles at zero)

Compute the inverse z-transform for $\quad W(z) = \dfrac{(z+1)}{z^2(z-1)}$

1. Re-write ($\ell = 2$) $\quad W(z) = c_0 + \dfrac{c_1}{z} + \dfrac{c_2}{z^2} + \dfrac{c_3 z}{(z-1)}$
2. Use the given formula to compute the constants :

$$
\begin{aligned}
c_2 &= \lim_{z\to 0}\left(z^2\frac{(z+1)}{z^2(z-1)}\right) = -1 \\
c_1 &= \lim_{z\to 0}\left(\frac{d}{dz}\left(z^2\frac{(z+1)}{z^2(z-1)}\right)\right) = -2 \\
c_0 &= \lim_{z\to 0}\left(\frac{1}{2}\frac{d^2}{dz^2}\left(z^2\frac{(z+1)}{z^2(z-1)}\right)\right) = -2 \\
c_3 &= \lim_{z\to 1}\left(\frac{z-1}{z}\frac{(z+1)}{z^2(z-1)}\right) = 2
\end{aligned}
$$

3. Finally : $w(k) = -2\Delta(k) - 2\Delta(k-1) - \Delta(k-2) + 2 \qquad k \geq 0$

3. Multiple non-zero poles :

$$W(z) = \cdots + \frac{c_1 z}{(z-p)} + \frac{c_2 z}{(z-p)^2} + \cdots + \frac{c_\ell z}{(z-p)^\ell} + \cdots$$

We have

$$\frac{(z-p)^\ell}{z} W(z) = \cdots + c_1 (z-p)^{\ell-1} + c_2 (z-p)^{\ell-2} + \cdots + c_\ell + \cdots$$

Therefore : $\quad c_\ell = \lim\limits_{z \to p} \dfrac{(z-p)^\ell}{z} W(z)$. Let's take the following derivative :

$$\begin{aligned} \frac{d}{dz}\left(\frac{(z-p)^\ell}{z} W(z)\right) \quad &= \quad \cdots + c_1(\ell-1)(z-p)^{\ell-2} + c_2(\ell-2)(z-p)^{\ell-2} \\ & \qquad + \cdots + 2c_{\ell-2}(z-p) + c_{\ell-1} + \cdots \end{aligned}$$

Therefore : $\quad c_{\ell-1} = \lim\limits_{z \to p} \dfrac{d}{dz}\left(\dfrac{(z-p)^\ell}{z} W(z)\right).$

3. Multiple non-zero poles (suit) : For the general case we have

$$c_{\ell-i} = \lim_{z\to p}\left(\frac{1}{i!}\frac{d^i}{dz^i}\left(\frac{(z-p)^\ell}{z}W(z)\right)\right) \qquad i = 0, 1, \ldots, \ell-1$$

From the Table we have

$$\mathcal{Z}^{-1}\left\{\frac{z}{(z-p)^2}\right\} = kp^{k-1} \quad \text{and} \quad \mathcal{Z}^{-1}\left\{\frac{z}{(z-p)^3}\right\} = \frac{1}{2}k(k-1)p^{k-2}$$

For the general case we have :

$$\mathcal{Z}^{-1}\left\{\frac{z}{(z-p)^\ell}\right\} = \frac{1}{(\ell-1)!}\left(\prod_{i=0}^{\ell-2}(k-i)\right)p^{k-\ell+1}$$

Therefore

$$w(k) = \cdots + c_1 p^k + c_2 kp^{k-1} + \cdots + \frac{c_\ell}{(\ell-1)!}\left(\prod_{i=0}^{\ell-2}(k-i)\right)p^{k-\ell+1} + \cdots \quad k \geq 0$$

Inverse z-Transform

Example (Multiple non-zero poles)

Compute the inverse z-transform for $W(z) = \dfrac{z^3 - 2z^2 + 2z}{(z-1)^2(z-2)}$

1. Re-write $W(z) = c_0 + \dfrac{c_1 z}{z-1} + \dfrac{c_2 z}{(z-1)^2} + \dfrac{c_3 z}{(z-2)}$
2. Take $c_0 = W(0) = 0$ and

$$\begin{aligned}
c_2 &= \lim_{z\to 1}\left(\frac{(z-1)^2}{z}\frac{z^3 - 2z^2 + 2z}{(z-1)^2(z-2)}\right) = -1 \\
c_1 &= \lim_{z\to 1}\left(\frac{d}{dz}\left(\frac{(z-1)^2}{z}\frac{z^3 - 2z^2 + 2z}{(z-1)^2(z-2)}\right)\right) = -1 \\
c_3 &= \lim_{z\to 2}\left(\frac{z-2}{z}\frac{z^3 - 2z^2 + 2z}{(z-1)^2(z-2)}\right) = 2
\end{aligned}$$

3. Finally : $w(k) = -1 - k + 2^{k+1} \qquad k \geq 0$

Inverse z-Transform

Alternate method to compute the constants :

Step 1 : Consider the expansion of $W(z)$:

$$W(z) = c_0 + c_1 V_1(z) + c_2 V_2(z) + \cdots$$

where $V_i(z) = \dfrac{z}{(z - p_i)^{n_i}}$. This expression is linear in $C = [c_0, c_1, \ldots, c_m]^T$.

Step 2 : Choose $z_0, z_1, \ldots, z_m$ different from the poles $z_j \neq p_i$.

Step 3 : Solve the following system of linear equations :

$$\begin{bmatrix} W(z_0) \\ W(z_1) \\ \vdots \\ W(z_m) \end{bmatrix} = \begin{bmatrix} 1 & V_1(z_0) & V_2(z_0) & \cdots \\ 1 & V_1(z_1) & V_2(z_1) & \cdots \\ 1 & \vdots & \vdots & \vdots \\ 1 & V_1(z_m) & V_2(z_m) & \cdots \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \end{bmatrix}$$

Inverse z-Transform

Theorem (Numerical inverse z-transform)

Let $W(z)$ be a rational function as :

$$W(z) = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

The inverse z-transform can be generated by the following recursive difference equation

$$w(k) = b_k - \sum_{\ell=1}^{n} a_\ell w(k-\ell)$$

where $b_i = 0$ for $i > n$.

Proof : From the rational function, we have :

$$b_0 z^n + b_1 z^{n-1} + \cdots + b_n = (z^n + a_1 z^{n-1} + \cdots + a_n) W(z)$$

Multiplying both sides by z^{-n}

$$b_0 + b_1 z^{-1} + \cdots + b_n z^{-n} = (1 + a_1 z^{-1} + \cdots + a_n z^{-n}) W(z)$$

Inverse z-Transform

Proof (suit) : Computing the inverse z-transform, we have :

$$b_0\Delta(k) + b_1\Delta(k-1) + \cdots + b_n\Delta(k-n) = w(k) + a_1 w(k-1) + \cdots + a_n w(k-n)$$

The left side is equal to b_k. So we have :

$$b_k = w(k) + \sum_{\ell=1}^{n} a_\ell w(k-\ell) \quad \Rightarrow \quad w(k) = b_k - \sum_{\ell=1}^{n} a_\ell w(k-\ell)$$

Example

Find the inverse z-transform of $W(z)$ by a recursive difference equation.

$$W(z) = \frac{z^3 - 2z^2 + 2z}{z^3 - 4z^2 + 5z - 2}$$

Solution : With $a_1 = -4, a_2 = 5, a_3 = -2, b_0 = 1, b_1 = -2$ and $b_2 = 2$:

$$w(0) = 1, w(1) = 2, w(2) = 5, w(3) = 12, w(4) = 27, \ldots$$

Question

Find using numerical inversion the inverse Z transform of

$$W(z) = \frac{z+3}{z^2 - 3z + 2}$$

(A) $\{w(k)\} = \{\ldots, \mathbf{1}, 6, 16, 36, \ldots\}$
(B) $\{w(k)\} = \{\ldots, \mathbf{0}, 1, 0, -2, -6, \ldots\}$
(C) $\{w(k)\} = \{\ldots, \mathbf{0}, 1, 6, 16, 26, \ldots\}$
(D) None of the above

Inverse z-Transform

Characteristics of the inverse z-transform of rational functions :

$$W(z) = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{(z - p_1)(z - p_2)\cdots(z - p_n)}$$

- The signal $w(k)$ is the weighted sum of signals based on the poles :

$$w(k) = c_0 \Delta(k) + c_1 p_1^k + \cdots + 2|c_i| r^k \cos(k\omega + \phi) + \cdots + c_n p_n^k$$

- if $|p| < 1$ then the corresponding signal converges to zero.
- if $|p| > 1$ then the corresponding signal diverges.
- If p real and $0 < p < 1$ then p^k converges monotonically to zero.
- If p is real and $-1 < p < 0$ then p^k converges non monotonically to zero (we have a chattering effect).

Discrete-Time Transfer Function

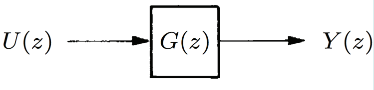
Convolution sum : The output of an LTI discrete-time system to the input signal $u(k)$ is given by : $y(k) = \sum_{\ell=0}^{k} u(\ell)g(k-\ell) = g(k) * u(k)$.

Taking the z-transform from the both sides gives : $Y(z) = G(z)U(z)$.

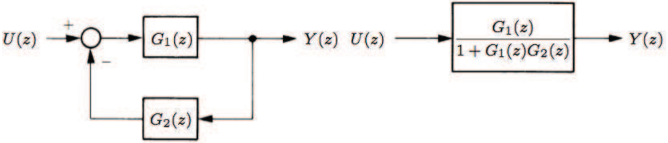
Discrete-Time Transfer Function

The z-transform of the impulse response $\{g(k)\}$ of a causal LTI discrete-time system is called its **discrete-time transfer function**.

$$\mathcal{Z}\{g(k)\} = G(z) = \frac{Y(z)}{U(z)}$$



The same rules as continuous-time case applied to closed-loop systems



Discrete-Time Transfer Function

Continuous-time and discrete-time systems are very similar

Continuous time System	Discrete-time System
$g(t)$: Response to a Dirac impulse	$g(k)$: Response to a unit impulse
Convolution integral $y(t) = \int_0^t u(\tau)g(t-\tau)d\tau$	Convolution sum $y(k) = \sum_{\ell=0}^{k} u(\ell)g(k-\ell)$
Differential equation $\frac{dy(t)}{d(t)} + ay(t) = bu(t)$	Difference equation $y(k) + cy(k-1) = du(k)$
Laplace Transform $Y(s) = \mathcal{L}(y(t))$ Transfer function in s $G(s) = \mathcal{L}(g(t))$ $Y(s) = G(s)U(s)$	z-Transform $Y(z) = \mathcal{Z}\{y(k)\}$ Transfer function in z $G(z) = \mathcal{Z}\{g(k)\}$ $Y(z) = G(z)U(z)$

Discrete-Time Transfer Function

Example (From difference equation to transfer function :)

Given the following difference equation

$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + \cdots + b_m u(k-m)$$

Find the transfer function between the input $U(z)$ and the output $Y(z)$.

Solution Take the z-transform of the both sides :

$$(1 + a_1 z^{-1} + \cdots + a_n z^{-n}) Y(z) = (b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}) U(z)$$

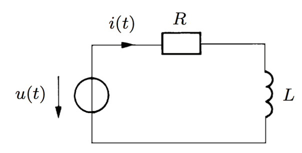
Therefore, (multiplying by z^n) :

$$G(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

Discrete-Time Transfer Function

Example

Compute a discrete-time transfer function :



$$L\frac{di(t)}{dt} + Ri(t) = u(t)$$

The derivative can be approximated by :

$$\frac{di(kh)}{dt} \approx \frac{i(kh) - i(kh-h)}{h} \quad \Rightarrow \quad (L+Rh)i(kh) - Li(kh-h) = hu(kh)$$

Taking the z-transform, we have : $(L+Rh)I(z) - Lz^{-1}I(z) = hU(z)$

$$\Rightarrow \quad G(z) = \frac{I(z)}{U(z)} = \frac{h}{L+Rh-Lz^{-1}} = \frac{hz}{(L+Rh)z-L}$$

Discrete-Time Transfer Function

Example (Discrete PD controller)

An ideal PD controller is given by :

$$u(kh) = K_p\left(e(kh) + T_d\frac{de(kh)}{dt}\right) \approx K_p\left(e(kh) + T_d\frac{e(kh) - e(kh-h)}{h}\right)$$

Taking the z-transform, we get :

$$U(z) = K_p\left(E(z) + \frac{T_d}{h}(1 - z^{-1})E(z)\right)$$

$$K(z) = \frac{U(z)}{E(z)} = K_p\left(1 + \frac{T_d}{h}(1 - z^{-1})\right) = \frac{K_p(h + T_d)z - K_p T_d}{hz}$$

Frequency Response (Discrete-time case)

Find the frequency response of $H(z)$ to a sinusoidal signal $u(k) = \sin \omega kh$.
Solution : We find first the response to $u(k) = e^{j\omega kh}$.

$$u(k) = e^{j\omega kh} = j \sin \omega kh + \cos \omega kh \quad \Rightarrow \quad U(z) = \frac{z}{z - e^{j\omega h}}$$

$$\Rightarrow \quad Y(z) = H(z)U(z) = c_0 + \frac{c_1 z}{z - p_1} + \cdots + \frac{c_n z}{z - p_n} + \frac{cz}{z - e^{j\omega h}}$$

where p_i are the distinct poles of $H(z)$. Taking the inverse z transform :

$$y(k) = c_0 \Delta(k) + c_1 p_1^k + \cdots + c_n p_n^k + \mathcal{Z}^{-1} \left\{ \frac{cz}{z - e^{j\omega h}} \right\}$$

If the system is stable then all $|p_i| < 1$, and

$$\lim_{k \to \infty} y(k) = \mathcal{Z}^{-1} \left\{ \frac{cz}{z - e^{j\omega h}} \right\} = c\, e^{j\omega kh}$$

where

$$c = \lim_{z \to e^{j\omega h}} \frac{z - e^{j\omega h}}{z} Y(z) = H(e^{j\omega h})$$

Frequency Response (Discrete-time case)

Therefore, the steady state response to $u(k) = \sin \omega kh$ is :

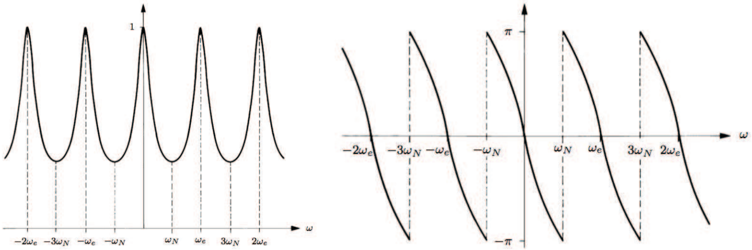
$$y_s(k) = I_m[H(e^{j\omega h})e^{j\omega kh}] = |H(e^{j\omega h})| \sin(\omega kh + \phi) \quad ; \quad \phi = \angle H(e^{j\omega h})$$

- $H(e^{j\omega h})$ is a periodic function with period $\omega_e = 2\pi/h$:

$$H(e^{j(\omega + \frac{2\pi}{h})h}) = H(e^{j\omega h}e^{j2\pi}) = H(e^{j\omega h})$$

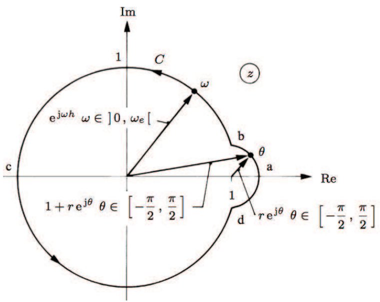
- $|H(e^{j\omega h})| = |H(e^{-j\omega h})|$ is an even function and $\angle H(e^{j\omega h}) = -\angle H(e^{-j\omega h})$ an odd function. They are usually computed for $\omega \in [0\,, \omega_N[$, where $\omega_N = \omega_e/2$ is the Nyquist frequency.

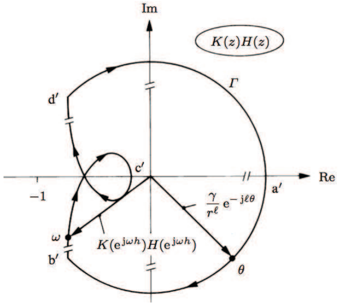
$|H(e^{j\omega h})|$ $\quad$ $\arg H(e^{j\omega h})$



Nyquist Stability Criterion

Discrete-time case : Let's consider the unit circle as the Nyquist contour Γ_z in the z-plane with a counterclockwise direction. It should avoid the poles on the imaginary axis (like integrators) by a very small detour. Its typical image under $K(z)H(z)$ is given below :





- The image of the detour $(z = 1 + re^{j\theta} \quad -90^\circ < \theta < 90^\circ)$, will be a semicircle with infinity radius.
- The image of the unit circle is the frequency response of $K(z)H(z) = K(e^{j\omega})H(e^{j\omega})$.

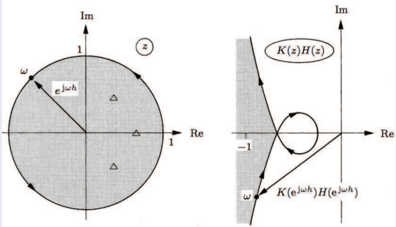
Nyquist Stability Criterion

Theorem (Discrete-time)

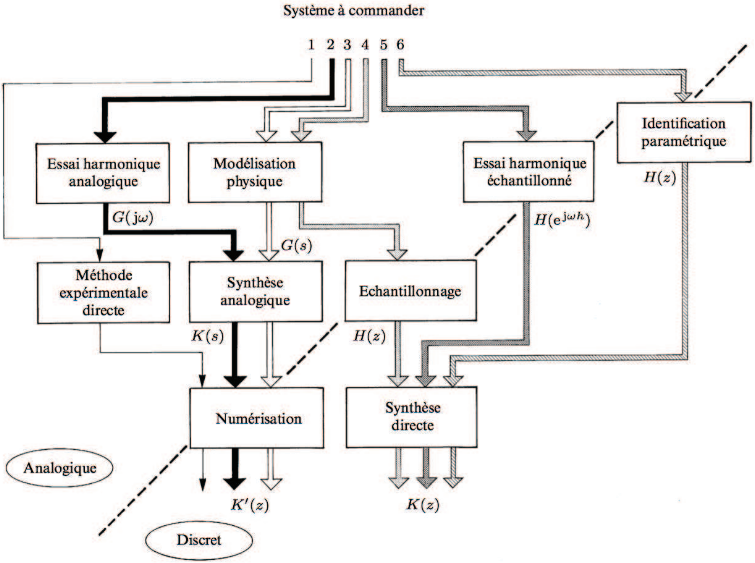
The zeros of $1 + K(z)H(z)$ are all inside the unit circle (i.e. the closed-loop system is stable), iff the image of Γ_z by the mapping $z \mapsto K(z)H(z)$ encircles counterclockwise the critical point (-1,0), P times, where P is the number of unstable poles of $K(z)H(z)$.

Simplified criterion

For open-loop transfer functions with no pole and zero outside the unit circle and maximum one pole on the unit circle, the closed-loop system is stable if the critical point is at the left side of the Nyquist plot, when ω goes from zero to ω_N.

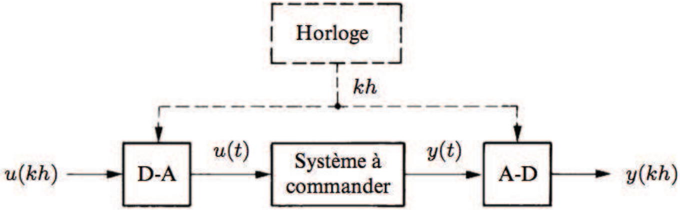


Continuous Versus Digital Controller Design

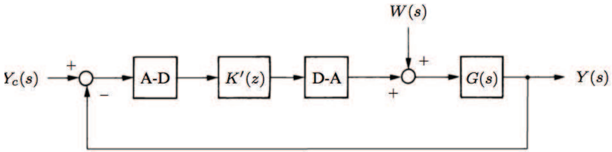


Discretization

Discretization of the plant model : The continuous-time plant model $G(s)$ is seen by the controller as a discrete system $H(z)$. Find $H(z)$ for direct digital controller synthesis.



Discretization of the Controller : A continuous-time controller $K(s)$ is designed. Find $K'(z)$ for implementation.

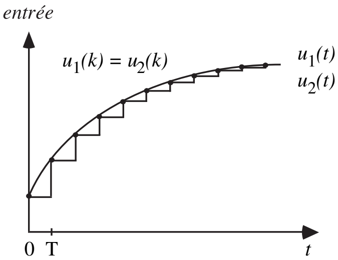


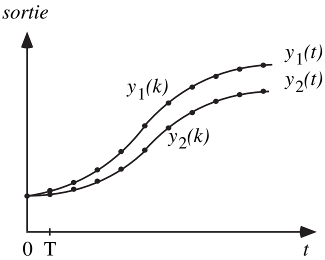
Discretization Methods :

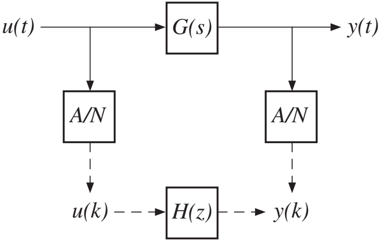
- No exact discretization method exists.
- Several methods lead to approximate discretization.
- For very small sampling periods, there is no significant difference between the approaches.

1. **Input-Output relation :**
    - Zero-Order-Hold method,
    - First-Order Hold method,
    - Impulse Response method.
2. **Derivative approximation :**
    - Euler approximation (forward method),
    - Euler approximation (backward method),
    - Bilinear approximation (Tustin).
3. **Zero-Pole matching**

Input-Output Relation







$$u_1(k) = u_2(k) \quad \Rightarrow \quad y_1(k) \neq y_2(k)$$

$$H_1(z) = \frac{Y_1(z)}{U_1(z)} \quad \neq \quad H_2(z) = \frac{Y_2(z)}{U_2(z)}$$

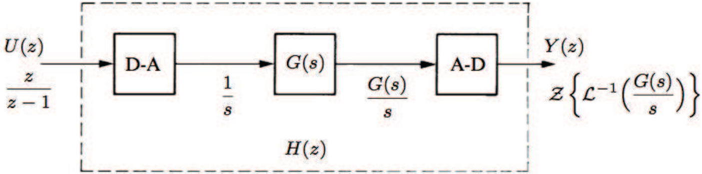
For a given $G(s)$, $H(z)$ depends on the input $u(t)$

Input-Output Relation

Principle : A transfer function is the relation between $Y(z)$ and $U(z)$. Therefore, for a choice of the input signal we have :

$$H(z) = \frac{Y(z)}{U(z)} = \frac{\mathcal{Z}\{y(k)\}}{\mathcal{Z}\{u(k)\}} = \frac{\mathcal{Z}\{\mathcal{L}^{-1}(G(s)U(s))\}}{\mathcal{Z}\{\mathcal{L}^{-1}(U(s))\}}$$

Zero-Order Hold method : The output of a ZOH is the weighted sum of shifted unit steps. So we consider a unit step as input ($U(s) = 1/s$) :



$$H(z) = \frac{\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\right\}}{\frac{z}{z-1}} = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\right\}$$

Input-Output Relation

Example

Find $H(z)$ from $G(s) = \dfrac{4}{s(s+2)}$ by the ZOH method ($h = 0.025$).

$$
\begin{aligned}
H(z) &= (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left[\frac{4}{s^2(s+2)}\right]\right\} \\
&= (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left[\frac{-1}{s} + \frac{2}{s^2} + \frac{1}{(s+2)}\right]\right\} \\
&= (1 - z^{-1})\left[\frac{-z}{z-1} + \frac{2hz}{(z-1)^2} + \frac{z}{z - e^{-2h}}\right] \\
&= \frac{(-1 + 2h + e^{-2h})z + 1 - e^{-2h}(1 + 2h)}{z^2 - (1 + e^{-2h})z + e^{-2h}} \\
&= \frac{10^{-3}(1.23z + 1.21)}{z^2 - 1.95z + 0.95} = \frac{10^{-3}(1.23z + 1.21)}{(z-1)(z-0.95)}
\end{aligned}
$$

Input-Output Relation

First-Order Hold method : In this case the input signal will be a unit ramp ($U(s) = 1/s^2$). This method is appropriate if a first-order hold is used after digital to analog conversion :

$$H(z) = \frac{Y(z)}{U(z)} = \frac{\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{G(s)}{s^2}\right)\right\}}{\frac{hz}{(z-1)^2}} = \frac{(z-1)^2}{hz}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{G(s)}{s^2}\right)\right\}$$

Impulse Response method : The input signal is a Dirac impulse ($U(s) = 1$). In this case $U(z)$ is not defined but can be approximated by :

$$U(z) = \mathcal{Z}\left\{\delta(t)\right\} = \lim_{h\to 0}\frac{1}{h}\mathcal{Z}\{\delta_h(t)\} \approx \frac{1}{h}$$

where $\delta_h(t)$ is a unit pulse with duration p. Therefore,

$$H(z) = \frac{Y(z)}{U(z)} \approx h\mathcal{Z}\{\mathcal{L}^{-1}(G(s))\}$$

Input-Output Relation

Systems with pure time delay : A continuous-time system with a pure time delay of T seconds in series has this transfer function : $G(s) = e^{-Ts}G'(s)$. If $T = dh$ where $d \in \mathbb{N}$ we obtain :

$$H(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left(e^{-dhs}\frac{G'(s)}{s}\right)\right\} = (1 - z^{-1})z^{-d}\mathcal{Z}\left\{\mathcal{L}^{-1}\left(\frac{G'(s)}{s}\right)\right\}$$

Example

The dynamic model of a physical system is given by

$$\dot{y}(t) + a\,y(t) = a\,u(t - T)$$

If $T = dh$, find its discrete transfer function using the ZOH method.

$$\frac{Y(s)}{U(s)} = e^{-sT}\frac{a}{s+a} = e^{-sT}G'(s) \quad \Rightarrow \quad \frac{G'(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

$$H(z) = (1 - z^{-1})z^{-d}\left(\frac{z}{z-1} - \frac{z}{z - e^{-ah}}\right) = \frac{1 - e^{-ah}}{z^d(z - e^{-ah})}$$

Derivative Approximation

Principle : Consider the following differential equation :

$$u(t) = \frac{d^n y(t)}{dt^n} \quad \Rightarrow \quad U(s) = s^n Y(s)$$

In discrete-time the above derivative can be approximated with different methods leading to different $H(z)$.

First method of Euler (forward) : For small sampling period h, we have

$$u(k) \approx \frac{y(k+1) - y(k)}{h} \quad \Rightarrow \quad U(z) = \frac{z-1}{h} Y(z)$$

So we can replace s in $G(s)$ with $\dfrac{z-1}{h}$ to find $H(z) = G(\frac{z-1}{h})$.

Example

Find $H(z)$ from $G(s) = \dfrac{4}{s(s+2)}$, where $h = 0.025$.

$$H(z) = \frac{4}{\frac{z-1}{h}\left(\frac{z-1}{h} + 2\right)} = \frac{4h^2}{(z-1)(z-1+2h)} = \frac{0.0025}{(z-1)(z-0.95)}$$

Derivative Approximation

Second method of Euler (backward) : For small sampling period h, we have

$$u(k) \approx \frac{y(k) - y(k-1)}{h} \quad \Rightarrow \quad U(z) = \frac{1 - z^{-1}}{h} Y(z)$$

So we can replace s in $G(s)$ with $\dfrac{z-1}{zh}$ to find $H(z) = G(\frac{z-1}{zh})$.

Example

Find $H(z)$ from $G(s) = \dfrac{4}{s(s+2)}$, where $h = 0.025$.

$$H(z) = \frac{4}{\frac{z-1}{zh}\left(\frac{z-1}{zh} + 2\right)} = \frac{4z^2h^2}{(z-1)(z-1+2zh)} = \frac{0.0025z^2}{(z-1)(1.95z-1)}$$

Derivative Approximation

Bilinear method (Tustin) : This method is based on the mean value of the forward and backward methods.

$$\begin{cases} \text{Forward :} & \frac{y(k+1)-y(k)}{h} = u(k) \\ \text{Backward :} & \frac{y(k)-y(k-1)}{h} = u(k) \Rightarrow \frac{y(k+1)-y(k)}{h} = u(k+1) \end{cases}$$

$$\frac{y(k+1)-y(k)}{h} = \frac{1}{2}[u(k)+u(k+1)] \quad \Rightarrow \quad \frac{z-1}{h}Y(z) = \frac{1}{2}(1+z)U(z)$$

So we can replace s in $G(s)$ with $\dfrac{2}{h}\dfrac{z-1}{z+1}$ to find $H(z) = G(\frac{2}{h}\frac{z-1}{z+1})$.

Example

Find $H(z)$ from $G(s) = \dfrac{4}{s(s+2)}$, using Tustin method.

$$H(z) = \frac{4}{\frac{2}{h}\frac{z-1}{z+1}\left(\frac{2}{h}\frac{z-1}{z+1}+2\right)} = \frac{h^2(z+1)^2}{(z-1)^2+h(z^2-1)}$$

Derivative Approximation

Relation between the poles of $G(s)$ and $H(z)$: How the left side of the imaginary axis in the s plane is mapped to the z plane.

Forward method

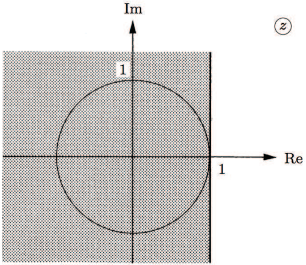
$$z = sh + 1$$

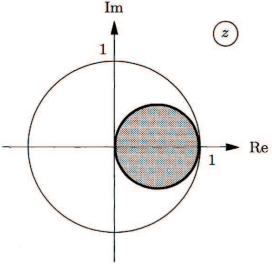
Backward method

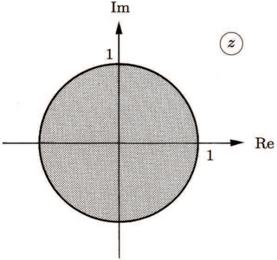
$$z = \frac{1}{1 - sh} = \frac{1}{2} + \frac{1}{2}\frac{1 + sh}{1 - sh}$$

Tustin method

$$z = \frac{1 + sh/2}{1 - sh/2}$$



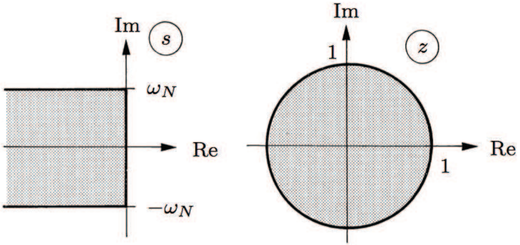




Remark : Forward approximation does not preserve the stability, i.e. a stable $G(s)$ can be transformed to an unstable $H(z)$.

Zero-Pole Matching

The function $z = e^{sh}$ maps the left hand side of the imaginary axis in the s plane to the inside of the unit circle in the z plane.



- $s = 0$ is mapped to $z = 1$ and $s = j\omega_N = j\pi/h$ is mapped to $z = e^{j\pi} = -1$.
- Relation with the derivative approximation methods :
  Forward approximation : $z = e^{sh} \approx 1 + sh$.
  Backward approximation : $z = e^{sh} = \frac{1}{e^{-sh}} \approx \frac{1}{1-sh}$.
  Bilinear approximation : $z = e^{sh} = \frac{e^{sh/2}}{e^{-sh/2}} \approx \frac{1+sh/2}{1-sh/2}$.

Zero-Pole Matching

Straightforward procedure to find $H(z)$ from $G(s)$:

1. Compute the poles p_i and zeros z_i of $G(s)$.
2. If the relative degree of $G(s)$ (degree of denominator minus degree of numerator) is greater than 1, consider some zeros at infinity such that relative degree becomes equal to 1.
3. Map the poles and zeros of $G(s)$ to the z plane :
   zeros of H(z) : $e^{z_i h}$ (each zero at infinity gives a zero at -1)
   poles of H(z) : $e^{p_i h}$
4. Compute the gain of $H(z)$ such that it has the same gain as $G(s)$ in a chosen frequency (usually at $s = 0$ to have the same steady state gain).

   $$\lim_{s \to 0} G(s) = \lim_{z \to 1} H(z)$$

   For systems with integrator, the steady-state gain is infinity so choose another frequency, e.g. the crossover frequency.

Zero-Pole Matching Method

Example

Given $G(s) = \dfrac{(s+1)}{(s+2)}$, find $H(z)$ using the zero-pole matching method.

Solution : We have one zero at -1 that leads to a zero at e^{-h} and a pole at -2 that gives a pole at e^{-2h}, therefore :

$$H(z) = \frac{c(z - e^{-h})}{(z - e^{-2h})}$$

We choose to have the steady state gain for both systems :

$$\lim_{s\to 0} G(s) = \lim_{z\to 1} H(z) \quad \Rightarrow \quad \frac{1}{2} = \frac{c(1 - e^{-h})}{(1 - e^{-2h})} \quad \Rightarrow \quad c = \frac{(1 - e^{-2h})}{2(1 - e^{-h})}$$

Zero-Pole Matching Method

Question

Given $G(s) = \dfrac{4}{s(s+2)}$ find $H(z)$ using the zero-pole matching method.

(A) $H(z) = \dfrac{c}{z(z - e^{-2h})}$ **(B)** $H(z) = \dfrac{c(z+1)}{(z-1)(z - e^{-2h})}$

(C) $H(z) = \dfrac{c}{(z - e^{-2h})}$ **(D)** $H(z) = \dfrac{c}{(z-1)(z - e^{-2h})}$

Match the gains at $\omega = 1$.

(A) $c = \infty$ **(B)** $c = (1 - e^{-2h})$

(C) $c = 2(1 - e^{-2h})$ **(D)** $c = \left| \dfrac{4(e^{jh} - 1)(e^{jh} - e^{-2h})}{\sqrt{5}(e^{jh} + 1)} \right|$

Frequency Response Plots (Discrete-time case)

Computing the frequency response : $H(e^{j\omega})$ can be computed by a fine grid of $\omega \in [0\,,\,\omega_N]$, where $\omega_N = \omega_e/2$ is the Nyquist frequency. Use `bode` and `nyquist` in Matlab to plot the Bode and Nyquist diagram.

Sketching the frequency response : Since $H(e^{j\omega})$ is not a polynomial function of ω, it cannot be easily plotted. However, $H(e^{j\omega}) \approx G(j\omega)$ in the frequency zone $\omega \in [0\,,\,\omega_N]$, where $G(s)$ is obtained by transformation of $H(z)$ to continuous time. The following methods are usually used :

- **Bilinear Transformation :** In this method we have

$$z = \frac{1 + sh/2}{1 - sh/2} \quad \Rightarrow \quad H(e^{j\omega}) \approx G(j\omega) = H\left(\frac{1 + \frac{j\omega h}{2}}{1 - \frac{j\omega h}{2}}\right)$$

- **ZOH method :** In this method we have

$$G(s) = s\mathcal{L}\left\{\mathcal{Z}^{-1}\left(\frac{z}{z-1}H(z)\right)\right\} \quad \Rightarrow H(e^{j\omega}) \approx G(j\omega)$$

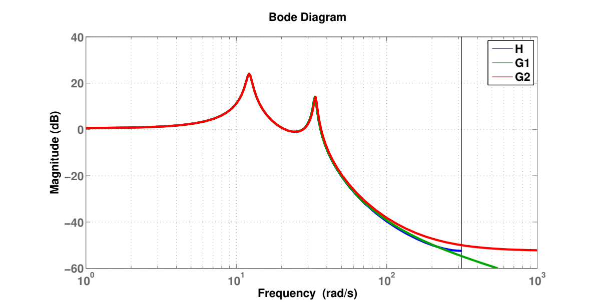
Frequency Response Plots (Discrete-time case)

Example

Consider $H(z)$ with $h = 0.01$ s :

$$H(z) = \frac{0.01184z^3 - 0.01645z^2 + 0.007212z - 0.0008898}{z^4 - 3.852z^3 + 5.683z^2 - 3.806z + 0.9762}$$

Compare the Bode plot of $H(e^{j\omega})$ and $G_1(j\omega)$ (ZOH transformation, green) and $G_2(j\omega)$ (Tustin transformation, red).



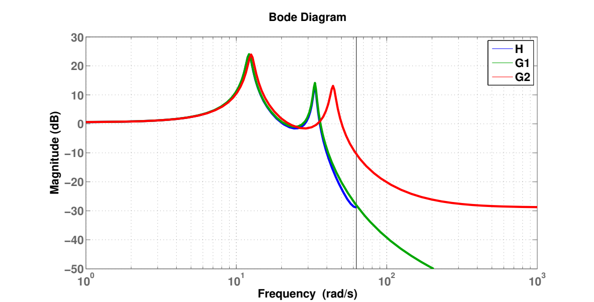
Frequency Response Plots (Discrete-time case)

Example

Consider the same dynamical system sampled with $h = 0.05$ s :

$$H(z) = \frac{0.2826z^3 + 0.5066z^2}{z^4 - 1.418z^3 + 1.589z^2 - 1.316z + 0.8864}$$

Compare the Bode plot of $H(e^{j\omega})$ and $G_1(j\omega)$ (ZOH transformation, green) and $G_2(j\omega)$ (Tustin transformation, red).



Find $G(s)$ from $H(z)$

The inverse of all methods can be used for conversion from discrete to continuous-time models.

ZOH method

$$G(s) = s\mathcal{L}\left\{\mathcal{Z}^{-1}\left(\frac{z}{z-1}H(z)\right)\right\}$$

impulse response

$$G(s) = \frac{1}{h}\mathcal{L}\left\{\mathcal{Z}^{-1}(H(z))\right\}$$

forward difference

$$G(s) = H(z)\Big|_{z=1+hs}$$

Tustin method

$$G(s) = H(z)\Big|_{z=\frac{1+hs/2}{1-hs/2}}$$

Conclusions of Discretizing Methods

- For very small sampling periods all methods give good approximation.
- The ZOH method is used for discretizing a plant model $G(s)$ if a zero order hold is used in the converters (almost always).
- For discretizing a controller $K(s)$, the backward difference method is usually used with a small sampling period (forward difference should not be used). Tustin gives a better approximation but is more complicated.
- The ZOH, Impulse and Zero-Pole Matching methods preserve the modes (poles) in transformation. The zeros are only preserved in the Zero-Pole Matching method.
- For converting $H(z)$ to $G(s)$ the Tustin method is usually used (backward difference should not be used). If $H(z)$ has high frequency resonance modes the ZOH or Zero-Pole matching may be preferred.

Digital Controller Design

- Discrete-Time Models
- RST Controller
- Pole Placement Technique
  - Desired Closed-loop Poles
  - Regulation (Diophantine Equation)
  - Tracking
- Related Design Methods
  - Internal Model Control (IMC)
  - Model Reference Control (MRC)

Difference Operator Models

We consider SISO-LTI discrete-time models of the form :

$$y(k) = -\sum_{i=1}^{n_A} a_i y(k-i) + \sum_{i=0}^{n_B} b_i u(k-i)$$

Define q^{-1} a backward shift operator such that $q^{-1}y(k) = y(k-1)$, then

$$A(q^{-1})y(k) = B(q^{-1})u(k)$$

$$\text{or} \quad y(k) = G(q^{-1})u(k) \quad \text{with} \quad G(q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})} \quad \text{where}$$

$$\begin{aligned} A(q^{-1}) &= 1 + a_1 q^{-1} + \cdots + a_{n_A} q^{-n_A} \\ B(q^{-1}) &= b_0 + b_1 q^{-1} + b_2 q^{-2} + \cdots + b_{n_B} q^{-n_B} \end{aligned}$$

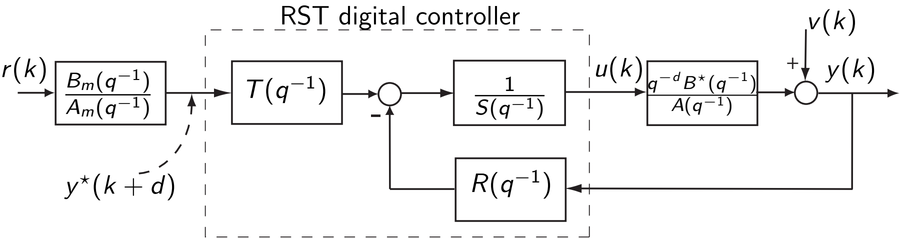
Delay d

The number of first zero coefficients of $B(q^{-1})$ is called delay d. For sampled systems $d \geq 1$ (b_0 is always zero) and $B(q^{-1}) = q^{-d}B^{\star}(q^{-1})$ where $B^{\star}(q^{-1})$ has no leading zero coefficients.

RST Controller

A general form of a two-degree of freedom digital controller is given by :

$$R(q^{-1})y(k) + S(q^{-1})u(k) = T(q^{-1})y^\star(k+d)$$



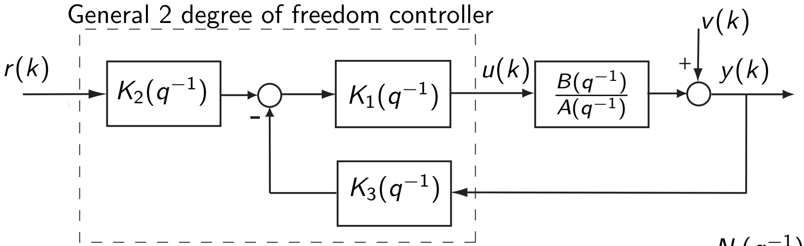
where $y^\star(k+d)$ is the *desired tracking trajectory* given with d steps in advance and

$$\begin{aligned} R(q^{-1}) &= r_0 + r_1 q^{-1} + \cdots + r_{n_R} q^{-n_R} \\ S(q^{-1}) &= 1 + s_1 q^{-1} + \cdots + s_{n_S} q^{-n_S} \\ T(q^{-1}) &= t_0 + t_1 q^{-1} + \cdots + t_{n_T} q^{-n_T} \end{aligned}$$

RST Controller

Advantages of RST controller :

- Can be easily implemented.
- It has two degrees of freedom (tracking and regulation dynamics can be designed independently).
- The other controller structures can be converted to an RST controller.



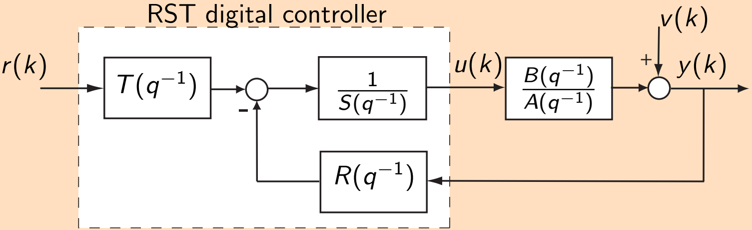
$$u(k) = K_1(q^{-1})\left[K_2(q^{-1})r(k) - K_3(q^{-1})y(k)\right] \quad \text{with} \quad K_i(q^{-1}) = \frac{N_i(q^{-1})}{D_i(q^{-1})}$$

is equivalent to

$$\begin{aligned} R(q^{-1}) &= N_1(q^{-1})N_3(q^{-1})D_2(q^{-1}) \\ S(q^{-1}) &= D_1(q^{-1})D_2(q^{-1})D_3(q^{-1}) \\ T(q^{-1}) &= N_1(q^{-1})N_2(q^{-1})D_3(q^{-1}) \end{aligned}$$

RST Controller

Question



1. What is the transfer function between r and y assuming $v(k) = 0$?

   (A) $\frac{T(q^{-1})B(q^{-1})}{A(q^{-1})S(q^{-1})+B(q^{-1})R(q^{-1})}$ **(B)** $\frac{A(q^{-1})R(q^{-1})}{A(q^{-1})S(q^{-1})+B(q^{-1})R(q^{-1})}$

   (C) $\frac{A(q^{-1})S(q^{-1})}{A(q^{-1})S(q^{-1})+B(q^{-1})R(q^{-1})}$ **(D)** $\frac{B(q^{-1})R(q^{-1})}{A(q^{-1})S(q^{-1})+B(q^{-1})R(q^{-1})}$

2. What is the transfer function between v and y assuming $r(k) = 0$?

   (A) $\frac{T(q^{-1})B(q^{-1})}{A(q^{-1})S(q^{-1})+B(q^{-1})R(q^{-1})}$ **(B)** $\frac{A(q^{-1})R(q^{-1})}{A(q^{-1})S(q^{-1})+B(q^{-1})R(q^{-1})}$

   (C) $\frac{A(q^{-1})S(q^{-1})}{A(q^{-1})S(q^{-1})+B(q^{-1})R(q^{-1})}$ **(D)** $\frac{B(q^{-1})R(q^{-1})}{A(q^{-1})S(q^{-1})+B(q^{-1})R(q^{-1})}$

Pole Placement Technique

Objective :

Place the closed-loop poles on the desired places.

Closed-loop poles

The roots of the characteristic polynomial $P(q^{-1})$ are the closed-loop poles.

$$P(q^{-1}) = A(q^{-1})S(q^{-1}) + B(q^{-1})R(q^{-1}) = 1 + p_1 q^{-1} + p_2 q^{-2} + \cdots$$

Desired closed-loop poles :

They should be chosen according to the desired performance.

Example (First-order polynomial)

Let $P(q^{-1}) = 1 + p_1 q^{-1}$. When $r(k) \equiv 0$, the free output response is defined by $y(k+1) = -p_1 y(k)$. Then $p_1 = -0.5$ leads to a relative decrease of 50% for the output amplitude at each sampling instant (choose p_1 between -0.2 and -0.8).

Pole Placement Technique

Example (Second-order polynomial)

Let $P(q^{-1}) = 1 + p_1 q^{-1} + p_2 q^{-2}$

1. Choose the time-domain performance (desired rise time, settling-time and overshoot for a step response).
2. Choose ζ (damping factor) and ω_n (natural frequency) of a second-order **continuous-time** model

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

   that meets the time-domain performance.
3. Compute s_1 and s_2, the roots of $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$.
4. Compute p_1 and p_2 from :

$$P(z^{-1}) = (z - e^{s_1 h})(z - e^{s_2 h}) = z^2 + p_1 z + p_2$$

Time-domain Performance

Overshoot : The overshoot M_p is a function of the damping factor :

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

Settling-time : The time t_s for which the response remains within 2% of the final value :

$$e^{-\zeta\omega_n t_s} < 0.02 \qquad \text{or} \qquad \zeta\omega_n t_s \approx 4$$

Rise-time : The time it takes to rise from 10% to 90% of the final value. The following approximation can be used :

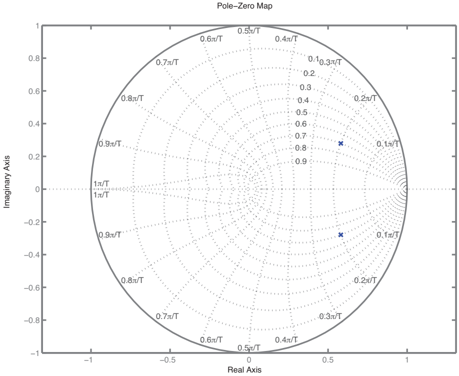
$$t_r \approx \frac{1.8}{\omega_n}$$

After computing ζ and ω_n, the desired $P(q^{-1})$ is computed by :

$$\begin{aligned} p_1 &= -2e^{-\zeta\omega_n h}\cos\left(\omega_n h\sqrt{1-\zeta^2}\right) \\ p_2 &= e^{-2\zeta\omega_n h} \end{aligned}$$

Time-domain Performance

Using `zgrid` of MATLAB :



Desired closed-loop poles with the loci for constant ζ and ω_n

The typical values for ζ and ω_n are :

$$\frac{0.25}{h} \leq \omega_n \leq \frac{1.5}{h} \quad ; \quad 0.7 \leq \zeta \leq 1$$

Time-domain Performance

Example

Compute the desired discrete-time closed-loop polynomial to have an overshoot of 10% and a settling time of $t_s = 1.2$ s. Suppose that the sampling period $h = 0.1s$.

1. For 10% overshoot we have : $e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.1 \quad \Rightarrow \quad \zeta \approx 0.6$
2. The natural frequency is computed as $\omega_n \approx \frac{4}{\zeta t_s} = 5.55$.
3. The coefficients of the characteristic polynomial are :

$$\begin{aligned} p_1 &= -2e^{-\zeta\omega_n h}\cos\left(\omega_n h\sqrt{1-\zeta^2}\right) = -1.294 \\ p_2 &= e^{-2\zeta\omega_n h} = 0.513 \end{aligned}$$

that corresponds to the following desired poles :

$$z_{1,2} = 0.647 \pm j0.308$$

Dominant and Auxiliary Poles

The desired closed-loop polynomial can be divided into two polynomials defining the dominant and auxiliary closed-loop poles :

$$P(q^{-1}) = P_d(q^{-1})\, P_f(q^{-1})$$

Dominant closed-loop poles : Define the main dynamics of the closed-loop system in regulation and are computed based on the desired time-domain performance.

Auxiliary closed-loop poles : They introduce a filtering action in certain frequency regions in order to

- reduce the effect of the measurement noise ;
- smooth the variations of the control signal ;
- improve the robustness.

As a general rule, the "auxiliary poles" (called also the "observer poles"), are faster than the "dominant poles". It means that the roots of $P_f(q^{-1})$ should have a real part smaller than those of $P_d(q^{-1})$.

Regulation : Computation of R and S

Once $P(q^{-1})$ is specified, in order to compute

$$\begin{aligned} R(q^{-1}) &= r_0 + r_1 q^{-1} + \cdots + r_{n_R} q^{-n_R} \\ S(q^{-1}) &= 1 + s_1 q^{-1} + \cdots + s_{n_S} q^{-n_S} \end{aligned}$$

the following equation, known as "Bezout identity" (or Diophantine equation), must be solved :

$$A(q^{-1})S(q^{-1}) + B(q^{-1})R(q^{-1}) = P(q^{-1})$$

Theorem

The Diophantine equation has a unique solution with minimal degree for

$$\begin{aligned} n_R &= deg\ R(q^{-1}) = n_A - 1 \\ n_S &= deg\ S(q^{-1}) = n_B - 1 \\ n_P &= deg\ P(q^{-1}) \neg \leq n_A + n_B - 1 \end{aligned}$$

If and only if $A(q^{-1})$ and $B(q^{-1})$ are coprime.

Regulation : Computation of R and S

Question

$$\text{Given} \qquad G(q^{-1}) = \frac{0.2q^{-2}}{1 - 0.8q^{-1}}$$

Compute minimum order of $R(q^{-1})$ and $S(q^{-1})$:

(A)	$n_R = 1, n_S = 1$	**(B)**	$n_R = 0, n_S = 0$
(C)	$n_R = 0, n_S = 1$	**(D)**	$n_R = 0, n_S = 2$

Regulation : Computation of R and S

Question

$$\text{Given} \qquad G(q^{-1}) = \frac{0.2q^{-2}}{1 - 0.8q^{-1}}$$

Compute $R(q^{-1}) = r_0$ and $S(q^{-1}) = 1 + s_1 q^{-1}$ to place the closed loop poles at the roots of $P(q^{-1}) = 1 - 1.3q^{-1} + 0.5q^{-2}$.

(A) $r_0 = 0.5, s_1 = -0.5$ **(B)** $r_0 = -0.5, s_1 = -0.5$

(C) $r_0 = -0.5, s_1 = 0.5$ **(D)** $r_0 = -5.9, s_1 = -2.1$

Regulation : Computation of R and S

Example

Consider a discrete-time plant model given by :

$$G(q^{-1}) = \frac{b_1 q^{-1} + b_2 q^{-2}}{1 + a_1 q^{-1} + a_2 q^{-2}} \qquad n_A = 2, \quad n_B = 2$$

Then $n_P \leq n_A + n_B - 1 = 3$, and $n_R = n_A - 1 = 1$, $n_S = n_B - 1 = 1$.
Let us take $n_P = 2$. Therefore, we should solve :

$$(1+a_1 q^{-1}+a_2 q^{-2})(1+s_1 q^{-1})+(b_1 q^{-1}+b_2 q^{-2})(r_0+r_1 q^{-1}) = 1+p_1 q^{-1}+p_2 q^{-2}$$

Then we have :

$$\begin{aligned} a_1 + s_1 + b_1 r_0 &= p_1 \\ a_2 + a_1 s_1 + b_2 r_0 + b_1 r_1 &= p_2 \\ a_2 s_1 + b_2 r_1 &= 0 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & b_1 & 0 \\ a_2 & a_1 & b_2 & b_1 \\ 0 & a_2 & 0 & b_2 \end{bmatrix} \begin{bmatrix} 1 \\ s_1 \\ r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} 1 \\ p_1 \\ p_2 \\ 0 \end{bmatrix}$$

Solving Diophantine Equation (Computing R and S)

A general solution to $A(q^{-1})S(q^{-1}) + B(q^{-1})R(q^{-1}) = P(q^{-1})$ is given by :

$$x = M^{-1}p$$

where $x^T = \begin{bmatrix} 1 & s_1 & \ldots & s_{n_S} & r_0 & \ldots & r_{n_R} \end{bmatrix}$ and

$$M = \begin{array}{c} \begin{matrix} \overbrace{\hphantom{1 \quad 0 \quad \cdots \quad 0}}^{n_B} & \overbrace{\hphantom{b_0 \quad 0 \quad \cdots \quad 0}}^{n_A} \end{matrix} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & 1 & \ddots & \vdots & b_1 & b_0 & \ddots & \vdots \\ \vdots & a_1 & \ddots & 0 & \vdots & b_1 & \ddots & 0 \\ a_{n_A} & \vdots & \ddots & 1 & b_{n_B} & \vdots & \ddots & b_0 \\ 0 & a_{n_A} & \ddots & a_1 & 0 & b_{n_B} & \ddots & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n_A} & 0 & \cdots & 0 & b_{n_B} \end{bmatrix} \end{array}$$

M is called the Sylvester matrix and $p^T = \begin{bmatrix} 1 & p_1 & \ldots & p_{n_P} & 0 & \ldots & 0 \end{bmatrix}$.
Note that the inverse of M exists if and only if $A(q^{-1})$ and $B(q^{-1})$ are coprime polynomials (no simplifications between zeros and poles).

Regulation : Computation of R and S

Fixed terms in the regulator : The performance and robustness of the closed-loop system can be improved by introducing some fixed terms, $H_R(q^{-1})$ and $H_S(q^{-1})$, in the polynomial R and S as :

$$\begin{aligned} R(q^{-1}) &= H_R(q^{-1})R'(q^{-1}) \\ S(q^{-1}) &= H_S(q^{-1})S'(q^{-1}) \end{aligned}$$

Therefore, we need to solve the following equation :

$$A(q^{-1})H_S(q^{-1})S'(q^{-1}) + B(q^{-1})H_R(q^{-1})R'(q^{-1}) = P(q^{-1})$$

This can be done after replacing $A(q^{-1})H_S(q^{-1})$ by $A'(q^{-1})$ and $B(q^{-1})H_R(q^{-1})$ by $B'(q^{-1})$.

Then, for the minimal order solution we should have :

$$\begin{aligned} n_{R'} &= \deg\ R'(q^{-1}) = n_{A'} - 1 = n_A + n_{H_S} - 1 \\ n_{S'} &= \deg\ S'(q^{-1}) = n_{B'} - 1 = n_B + n_{H_R} - 1 \\ n_P &= \deg\ P(q^{-1}) \leq n_{A'} + n_{B'} - 1 = n_A + n_{H_S} + n_B + n_{H_R} - 1 \end{aligned}$$

Regulation : Choice of H_R and H_S

Choice of H_S

- Zero steady state error for a step disturbance :
  Integrator in the controller : $H_S = 1 - q^{-1}$
- Asymptotic rejection of a harmonic disturbance $v(k)$:

$$v(k) = \frac{1}{1 + \alpha q^{-1} + q^{-2}} \delta(k) \qquad \alpha = -2\cos(\omega h) = -2\cos(2\pi f h)$$

  Internal model principle : $H_S(q^{-1}) = 1 + \alpha q^{-1} + q^{-2}$

Choice of H_R

- opening the loop ($u = 0$) at a disturbance frequency f :

$$H_R(q^{-1}) = (1 + \alpha q^{-1} + q^{-2})$$

- Opening the loop at Nyquist frequency ($f = f_s/2 = 1/(2h)$) :

$$H_R(q^{-1}) = 1 + q^{-1}$$

Computation of T

The tracking performance is usually given by a tracking reference model :

$$H_m(q^{-1}) = \frac{B_m(q^{-1})}{A_m(q^{-1})}$$

The transfer function from the reference to the output is :

$$H_{cl}(q^{-1}) = \frac{B_m(q^{-1})T(q^{-1})B(q^{-1})}{P(q^{-1})A_m(q^{-1})}$$

Different dynamic for regulation and tracking :

In this case, $T(q^{-1}) = P(q^{-1})/B(1)$ cancels the regulation dynamic and make the steady-state gain of $H_{cl}(q^{-1})$ equal to 1. The tracking dynamic is imposed by the denominator of the reference model.

Same dynamic for regulation and tracking :

The reference model is chosen as $H_m(q^{-1}) = 1$ and $T(q^{-1})$, is chosen to have a steady-state gain of 1 for $H_{cl}(q^{-1})$. So we take : $T(q^{-1}) = P(1)/B(1)$. If the controller or the plant model has an integrator, i.e. $A(1)S(1) = 0$: Then $P(1) = A(1)S(1) + B(1)R(1) = B(1)R(1)$ and $T(q^{-1}) = R(1)$.

RST controller design

Example

Consider the following discrete-time second-order plant model :

$$G(q^{-1}) = \frac{0.1q^{-1} + 0.2q^{-2}}{1 - 1.3q^{-1} + 0.42q^{-2}}$$

The sampling period is $h = 1s$.

Design an RST controller such that :

- The tracking dynamics are close to the dynamics of a second-order continuous-time model with $\omega_n = 0.5$ rad/s and $\zeta = 0.9$.
- The regulation dynamics are close to that of a second-order continuous-time model with $\omega_n = 0.4$ rad/s and $\zeta = 0.9$.
- The steady state error for an output step disturbance is zero.

RST controller design

Example

1. With $\omega_n = 0.4$ rad/s and $\zeta = 0.9$, we obtain :
$$P(q^{-1}) = 1 - 1.3741q^{-1} + 0.4867q^{-2}$$
2. Zero steady state error is obtained by $H_S(q^{-1}) = 1 - q^{-1}$.
3. The following Bezout equation should be solved :
$$A(q^{-1})H_S(q^{-1})S'(q^{-1}) + B(q^{-1})R(q^{-1}) = P(q^{-1})$$
We have $n_{S'} = n_B - 1 = 1$ and $n_R = n_A + n_{H_S} - 1 = 2$ and
$$A'(q^{-1}) = A(q^{-1})(1 - q^{-1}) = 1 - 2.3q^{-1} + 1.72q^{-2} - 0.42q^{-3}$$
Therefore the Bezout equation in the matrix form becomes :
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2.3 & 1 & 0.1 & 0 & 0 \\ 1.72 & -2.3 & 0.2 & 0.1 & 0 \\ -0.42 & 1.72 & 0 & 0.2 & 0.1 \\ 0 & -0.42 & 0 & 0 & 0.2 \end{bmatrix} \begin{bmatrix} 1 \\ s'_0 \\ r_0 \\ r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1.3741 \\ 0.4867 \\ 0 \\ 0 \end{bmatrix}$$

RST controller design

Example

Solving the Bezout equation leads to

$$\begin{aligned} R(q^{-1}) &= 3 - 3.94q^{-1} + 1.3141q^{-2} \\ S(q^{-1}) &= (1 + s_1' q^{-1})(1 - q^{-1}) = 1 - 0.3742q^{-1} - 0.6258q^{-2} \end{aligned}$$

4 : The reference model $H_m(q^{-1})$ is computed by discretization of a second-order model with $\omega_n = 0.5$ rad/s and $\zeta = 0.9$:

$$H_m(q^{-1}) = \frac{0.0927q^{-1} + 0.0687q^{-2}}{1 - 1.2451q^{-1} + 0.4066q^{-2}}$$

Finally, the polynomial $T(q^{-1})$ is computed as :

$$T(q^{-1}) = \frac{P(q^{-1})}{B(1)} = 3.333 - 4.5806q^{-1} + 1.6225q^{-2}$$

If we wish to have the same dynamics for tracking and regulation, then $H_m(q^{-1}) = 1$ and $T(q^{-1}) = R(1) = 0.3741$.

Internal Model Control (IMC)

IMC method is a special case of the pole placement technique.

- The plant poles are chosen as the dominant closed-loop poles : $P_d(q^{-1}) = A(q^{-1})$
- The plant model should be stable with well-damped poles.
- This technique has good robustness with respect to model uncertainty and appropriate control input.

The following equation should be solved :

$$A(q^{-1})S(q^{-1}) + B(q^{-1})R(q^{-1}) = A(q^{-1})P_f(q^{-1})$$

$$\Rightarrow \quad R(q^{-1}) = A(q^{-1})R'(q^{-1})$$

After elimination of the common factor $A(q^{-1})$:

$$S(q^{-1}) + B(q^{-1})R'(q^{-1}) = P_f(q^{-1})$$

Taking $R'(q^{-1}) = r_0'$ leads to a set of minimal order solutions :

$$S(q^{-1}) = P_f(q^{-1}) - B(q^{-1})r_0'$$

IMC with integrator : if we assume that $S(q^{-1})$ contains an integrator, i.e. $S(1) = 0$. Therefore, we have :

$$P_f(1) - B(1)r'_0 = S(1) = 0 \quad \Rightarrow r'_0 = \frac{P_f(1)}{B(1)}$$

as the simplest solution yielding :

$$R(q^{-1}) = A(q^{-1})\frac{P_f(1)}{B(1)}$$

$$S(q^{-1}) = P_f(q^{-1}) - \frac{B(q^{-1})P_f(1)}{B(1)}$$

For tracking we have :

$$T(q^{-1}) = \frac{A(q^{-1})P_f(q^{-1})}{B(1)}$$

or if we choose the same dynamics for tracking and regulation

$$T(q^{-1}) = \frac{A(1)P_f(1)}{B(1)}$$

Internal Model Control (IMC)

Question

Consider the following discrete-time second-order plant model :

$$G(q^{-1}) = \frac{0.2q^{-1} + 0.3q^{-2}}{1 - 1.3q^{-1} + 0.4q^{-2}}$$

Design an RST controller with integrator and the same dynamics for tracking and regulation based on IMC technique ($P_f = 1$).

(A)
$R(q^{-1}) = 5 - 6.5q^{-1} + 2q^{-2}$
$S(q^{-1}) = 1 - q^{-1} - 1.5q^{-2}$
$T(q^{-1}) = 0.5$

(B)
$R(q^{-1}) = 0.2 - 0.26q^{-1} + 0.08q^{-2}$
$S(q^{-1}) = 1 - 0.04q^{-1} - 0.06q^{-2}$
$T(q^{-1}) = 0.02$

(C)
$R(q^{-1}) = 2(1 - 1.3q^{-1} + 0.4q^{-2})$
$S(q^{-1}) = 1 - 0.4q^{-1} - 0.6q^{-2}$
$T(q^{-1}) = 0.5$

(D)
$R(q^{-1}) = 2 - 2.6q^{-1} + 0.8q^{-2}$
$S(q^{-1}) = 1 - 0.4q^{-1} - 0.6q^{-2}$
$T(q^{-1}) = 0.2$

Model Reference Control (MRC)

In this approach the zeros of the plant model in

$$H_{cl}(q^{-1}) = \frac{q^{-d}B^{\star}(q^{-1})}{A(q^{-1})S(q^{-1}) + q^{-d}B^{\star}(q^{-1})R(q^{-1})}$$

are cancelled by the closed-loop poles :

$$A(q^{-1})S(q^{-1}) + q^{-d}B^{\star}(q^{-1})R(q^{-1}) = B^{\star}(q^{-1})P(q^{-1})$$

This can be done if

- the zeros of $B^{\star}(q^{-1})$ are stable,
- complex zeros have a sufficiently high damping factor ($\zeta > 0.2$).

Remark

In discrete-time systems, unstable zeros can be the consequence of too fast sampling or a large fractional delay. This can be avoided by re-identification of a model with augmented delay or resampling.

Model Reference Control (MRC)

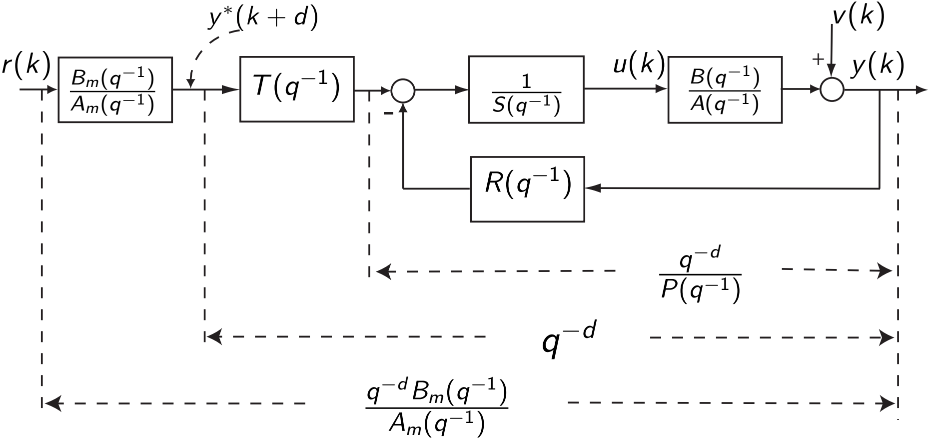
In order to have a solution to Diophantine Equation we should have $S(q^{-1}) = B^{\star}(q^{-1})S'(q^{-1})$ and solve

$$A(q^{-1})S'(q^{-1}) + q^{-d}R(q^{-1}) = P(q^{-1})$$

with $\quad n_P \leq n_A + d - 1 \quad , \quad n_{S'} = d - 1 \quad , \quad n_R = n_A - 1$ and

$$M = \begin{array}{c} \begin{matrix} \overbrace{\hphantom{1 \quad\quad 0 \quad\quad \cdots \quad\quad 0}}^{d} & \overbrace{\hphantom{0 \quad\; 0 \quad\; \cdots \quad\; 0}}^{n_A} \end{matrix} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & \ddots & \vdots & 0 & 0 & \ddots & \vdots \\ a_2 & a_1 & \ddots & 0 & \vdots & 0 & \ddots & 0 \\ \vdots & a_2 & \ddots & 1 & 0 & \vdots & \ddots & 0 \\ a_{n_A} & \vdots & \ddots & a_1 & 1 & 0 & \ddots & 0 \\ 0 & a_{n_A} & \ddots & a_2 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n_A} & 0 & \cdots & 0 & 1 \end{bmatrix} \end{array}$$

Model Reference Control (MRC)



Tracking : By choosing $T(q^{-1}) = P(q^{-1})$ the transfer function between the reference $r(k)$ and $y(k)$ will be :

$$H_{cl}(q^{-1}) = \frac{q^{-d}B_m(q^{-1})}{A_m(q^{-1})}$$

Model Reference Control

Example

Consider the following discrete-time second-order plant model :

$$G(q^{-1}) = \frac{0.2q^{-2} + 0.1q^{-3}}{1 - 1.3q^{-1} + 0.42q^{-2}}$$

Design an RST controller based on MRC technique for placing the closed loop dominant pole at 0.7.

- $B^\star(q^{-1}) = 0.2 + 0.1q^{-1}$ has a zero at -0.5 (inside the unit circle).
- Taking $P_d = 1 - 0.7q^{-1}$ and solving $AS + q^{-d}B^\star R = P_d B^\star$ gives $S = B^\star S'$. So we should solve $AS' + q^{-d}R = P_d$
- We have $n_{S'} = d - 1 = 1$ and $n_R = n_A - 1 = 1$ so we should solve :

$$(1 - 1.3q^{-1} + 0.42q^{-2})(1 + s_1'q^{-1}) + q^{-2}(r_0 + r_1q^{-1}) = 1 - 0.7q^{-1}$$

Example

We should solve :

$$(1-1.3q^{-1}+0.42q^{-2})(1+s_1'q^{-1})+q^{-2}(r_0+r_1q^{-1})=1-0.7q^{-1}$$

by the Sylvester Matrix method :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1.3 & 1 & 0 & 0 \\ 0.42 & -1.3 & 1 & 0 \\ 0 & 0.42 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ s_1' \\ r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.7 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -1.3+s_1' &= -0.7 & \Rightarrow s_1' = 0.6 \\ 0.42-1.3s_1'+r_0 &= 0 & \Rightarrow r_0 = 0.36 \\ 0.42s_1'+r_1 &= 0 & \Rightarrow r_1 = -0.252 \end{aligned}$$

Finally we compute : $T(q^{-1})=P_d(q^{-1})=1-0.7q^{-1}$.

Robust Pole Placement

A pole placement controller may not be implemented on the real system for the following reasons :

1. The controller may not have good robustness margins
   - Robustness can be verified using the robustness margins like gain, phase and modulus margin M_m (the inverse of the infinity norm of the sensitivity function). $M_m \geq 0.5$ implies a gain margin of greater than 2 and a phase margin of greater than $29°$.

$$\|\mathcal{S}\|_\infty = \left\| \frac{AS}{AS + q^{-d}BR} \right\|_\infty = \max_\omega |\mathcal{S}(e^{-j\omega})| < 6\text{dB} \quad \equiv \quad M_m > 0.5$$

2. The control input may be too large and saturated in real experiment.
   - The magnitude of the transfer function between the external input and the control input should be reduced at high frequencies.
   - The dominant closed loop poles should be slowed down.

Robust Pole Placement

Example

Consider the following plant model with $h = 1s$:

$$G(q^{-1}) = \frac{q^{-1} + 0.5q^{-2}}{1 - 1.5q^{-1} + 0.7q^{-2}}$$

- Desired closed-loop poles : $z_{1,2} = 0.3 \pm j0.2$.
- Integrator in the controller : $H_S(q^{-1}) = 1 - q^{-1}$.

Solving the Bezout equation, we obtain :

$$\begin{aligned} R(q^{-1}) &= 1.4667 - 1.72q^{-1} + 0.6067q^{-2} \\ S(q^{-1}) &= 1 - 0.5667q^{-1} - 0.4333q^{-2} \end{aligned}$$

This controller gives :

- $M_m = \|\mathcal{S}\|_\infty^{-1} = 0.39 \quad ; \quad \|\mathcal{U}\|_\infty \approx 17$ dB.
- $|u(k)| > 2$ for an impulse output disturbance.
- Settling time of the output disturbance step response : 6 sec

Robust Pole Placement

Example

Slowing down the closed loop poles :
The dominant poles of the plant model have $\omega_n = 0.4926$ and $\zeta = 0.362$. We choose the same ω_n with $\zeta = 0.9$ to compute the desired closed-loop poles ($z_{1,2} = 0.6272 \pm j0.1368$).
Solving the Bezout equation, we obtain :

$$\begin{aligned} R(q^{-1}) &= 0.8721 - 1.29q^{-1} + 0.5231q^{-2} \\ S(q^{-1}) &= 1 - 0.6264q^{-1} - 0.3736q^{-2} \end{aligned}$$

This controller gives :

- $M_m = \|\mathcal{S}\|_\infty^{-1} = 0.566 \quad ; \quad \|\mathcal{U}\|_\infty \approx 10$ dB.
- $|u(k)| < 1.5$ for an impulse output disturbance.
- Settling time of the output disturbance step response : 12 sec

The new controller is more robust but it's slower.

Robust Pole Placement

Example

Shaping the input sensitivity function :
We add a fixed term $H_R(q^{-1}) = 1 + q^{-1}$ in the controller to reduce the input sensitivity function at high frequencies but we keep the same closed-loop poles as the original controller (fast poles). This leads to :

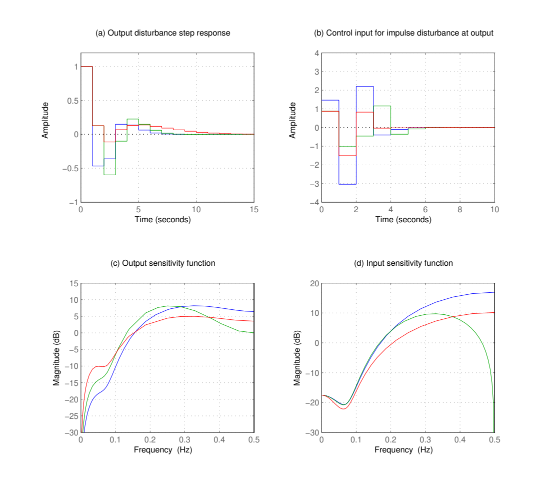
$$\begin{aligned} R(q^{-1}) &= 0.8740 - 0.2382q^{-1} - 0.6973q^{-2} + 0.4149q^{-3} \\ S(q^{-1}) &= 1 + 0.0260q^{-1} - 0.7297q^{-2} - 0.2964q^{-3} \end{aligned}$$

This controller gives :

- $M_m = \|\mathcal{S}\|_\infty^{-1} = 0.3913 \quad ; \quad \|\mathcal{U}\|_\infty \approx 10$ dB.
- $|u(k)| < 1$ for an impulse output disturbance.
- Settling time of the output disturbance step response : 7 sec

Robust Pole Placement

Example



Original controller (blue curves), slowing down the closed-loop poles (red curves), adding a fixed term $H_R(q^{-1}) = 1 + q^{-1}$ (green curves)