State Space Methods

Control Systems

Fall 2024

Analysis and synthesis of feedback control systems

Transform Methods (classical control)

- Use Laplace transform (or Z-transform) for analysis and synthesis.
- The plant and the controller are represented by transfer functions.
- The closed-loop performance are defined in the frequency domain (bandwidth, gain margin, phase margin, modulus margin, desired open-loop transfer function, reference model, etc.)
- Appropriate for industrial PID controller design.

State-Space Methods (modern control)

- Use the state space for analysis and synthesis.
- The plant model and the controller are represented in the state space.
- The closed-loop performance is defined in time-domain.
- It is appropriate for multi-input multi-output systems.
- It can be used for nonlinear systems.

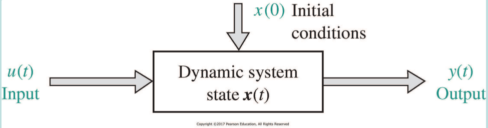
Outline

- **State-Space Models**
  - State-space model of physical systems
  - From transfer function to state-space model
  - Controllability and observability
  - From state-space model to transfer function
  - Stability of state-space models
- **Full-State Feedback Control**
  - Pole Placement design
  - Linear Quadratic Regulator design
  - Introducing the reference signal
  - Integral control
- **Estimator (Observer) Design**
- **Combining Control and Observer Design**

State-Space Models

State

The state of a system is a set of variables whose values, together with the input signals and the equations describing the dynamics, will provide the future state and output of the system.



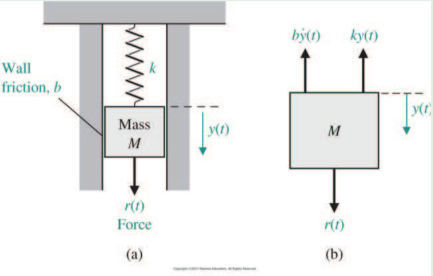
Example

- For mass-spring-damper systems, the states are usually the position and the velocity of the mass.
- For RLC circuits, the states are the capacitor voltage and the inductor current.
- In general, in a dynamic system represented by a differential equation, the initial conditions are related to the state variables.

State-Space Models

Example (Mass-Spring-Damper)

$$r(t) = Ma(t) + bv(t) + ky(t)$$



States :

$$x_1(t) = y(t)$$
$$x_2(t) = v(t)$$

State-Space Equations : Keep the derivative of the states in one side.

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{k}{M}x_1(t) - \frac{b}{M}x_2(t) + \frac{1}{M}r(t)$$
$$y(t) = x_1(t)$$

State-Space Models

Matrix form of state-space equations

The state space equations can be represented in matrix form as :

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + Du(t)$$

where $\mathbf{x}(t) \in \mathbb{R}^{n\times 1}$ is the state vector, $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the state matrix, $\mathbf{B} \in \mathbb{R}^{n\times 1}$, $\mathbf{C} \in \mathbb{R}^{1\times n}$.

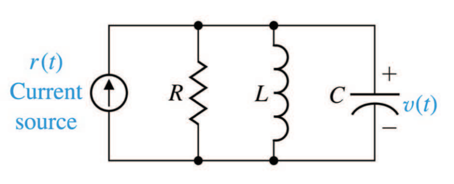
Example

The state-space equations for the mass-spring-damper is given as :

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{M} & \frac{-b}{M} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} r(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Example (RLC Circuit)

$$r(t) = \frac{1}{R}v(t) + C\frac{dv(t)}{dt} + i_L(t)$$



States :

$$x_1(t) = v(t)$$
$$x_2(t) = i_L(t)$$

State-Space Equations : Keep the derivative of the states in one side.

$$\dot{x}_1(t) = -\frac{1}{RC}x_1(t) - \frac{1}{C}x_2(t) + \frac{1}{C}r(t)$$
$$\dot{x}_2(t) = \frac{1}{L}x_1(t)$$
$$y(t) = x_1(t)$$

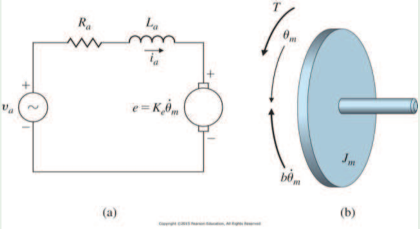
$$\mathbf{A} = \begin{bmatrix} \frac{-1}{RC} & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0$$

Example (DC Motor)

$$v_a = R_a i_a + L_a \frac{di_a}{dt} + K_e \dot{\theta}_m$$

$$K_t i_a = J_m \ddot{\theta}_m + b\dot{\theta}_m$$



States :

$$\mathbf{x}(t) = \begin{bmatrix} \theta_m(t) & \omega_m(t) & i_a(t) \end{bmatrix}^T$$

State-Space Equations :

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{b}{J_m}x_2(t) + \frac{K_t}{J_m}x_3(t)$$

$$\dot{x}_3(t) = -\frac{K_e}{L_a}x_2(t) - \frac{R_a}{L_a}x_3(t) + \frac{1}{L_a}v_a(t)$$

$$y(t) = x_1(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{-b}{J_m} & \frac{K_t}{J_m} \\ 0 & \frac{-K_e}{L_a} & \frac{-R_a}{L_a} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L_a} \end{bmatrix}$$

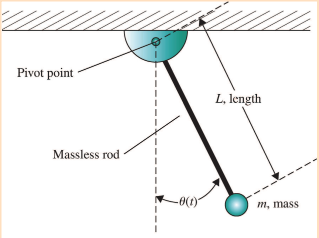
$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad D = 0$$

Exercise

For a simple pendulum, the linearized equation of motion is :

$$\ddot{\theta}(t) + \frac{g}{L}\theta(t) + \frac{k}{m}\dot{\theta}(t) = 0$$

Obtain a state space representation (output is $\theta(t)$).



(A) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -g/L & -k/m \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$

(B) $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ g/L & k/m \end{bmatrix}$ $\mathbf{B} = \mathbf{0}$ $\mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$

(C) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -g/L & -k/m \end{bmatrix}$ $\mathbf{B} = \mathbf{0}$ $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

(D) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -g/L & -k/m \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

Example (From TF model to SS model)

Consider the following transfer function :

$$G(s) = \frac{Y(s)}{U(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6}$$

Find a state-space equivalent representation of $G(s)$.

Solution : We have : $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$. Let's take $x_1 = \ddot{y}, x_2 = \dot{y}$ and $x_3 = y$. Then :

$$\begin{aligned} \dot{x}_1 &= -6x_1 - 11x_2 - 6x_3 + 6u \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= x_2 \\ y &= x_3 \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \qquad D = 0$$

State-Space Models

Example (From TF model to SS model)

Find a state-space equivalent representation of $G(s)$.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+7s+12}$$

Solution :

$$\Rightarrow Y(s) = \frac{sU(s)}{s^2+7s+12} + \frac{2U(s)}{s^2+7s+12} = sY_1(s) + 2Y_1(s)$$

We have : $\ddot{y}_1 + 7\dot{y}_1 + 12y_1 = u$. Let's take $x_1 = \dot{y}_1, x_2 = y_1$. Therefore, $y = \dot{y}_1 + 2y_1 = x_1 + 2x_2$

$$\begin{aligned} \dot{x}_1 &= -7x_1 - 12x_2 + u \\ \dot{x}_2 &= x_1 \\ y &= x_1 + 2x_2 \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 2 \end{bmatrix} \qquad D = 0$$

State-Space Models

Control Canonical Form

The state space representation of the following transfer function

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n}$$

in control canonical form is given by :

$$\mathbf{A}_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \end{bmatrix} \qquad \mathbf{B}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\mathbf{C}_c = \begin{bmatrix} b_1 & b_2 & \cdots & \cdots & b_n \end{bmatrix} \qquad D_c = 0$$

Example (Modal Form)

Find a state-space equivalent representation of $G(s)$ in Modal Form :

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+7s+12} = \frac{2}{s+4} + \frac{-1}{s+3}$$

$$\Rightarrow Y(s) = \frac{2U(s)}{s+4} + \frac{-U(s)}{s+3} = 2Y_1(s) - Y_2(s)$$

We have : $\dot{y}_1 = -4y_1 + u$ and $\dot{y}_2 = -3y_2 + u$. Let's take $x_1 = y_1$ and $x_2 = y_2$. Therefore, $y = 2x_1 - x_2$.

$$\begin{aligned} \dot{x}_1 &= -4x_1 + u \\ \dot{x}_2 &= -3x_2 + u \\ y &= 2x_1 - x_2 \end{aligned} \qquad \mathbf{A}_m = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix} \quad \mathbf{B}_m = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{C}_m = \begin{bmatrix} 2 & -1 \end{bmatrix} \qquad D_m = 0$$

Remark : $\mathbf{A}_m$ is a diagonal matrix of the poles of $G(s)$, $\mathbf{B}_m$ is a column vector of all 1, $\mathbf{C}_m$ is a row vector of the poles' residues and $D_m = 0$.

State-Space Models

Example (From TF model to SS model)

Find another state-space equivalent representation of $G(s)$.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+7s+12}$$

Solution :

$$\Rightarrow s^2Y(s) = s[U(s) - 7Y(s)] + [2U(s) - 12Y(s)]$$

$$\Rightarrow sY(s) = [U(s) - 7Y(s)] + \frac{1}{s}[2U(s) - 12Y(s)]$$

Let's take $x_1 = y$ and $\dot{x}_2 = 2u - 12y$. Therefore, $\dot{y} = u - 7y + x_2$.

$$\dot{x}_1 = -7x_1 + x_2 + u$$
$$\dot{x}_2 = -12x_1 + 2u$$
$$y = x_1$$

$$\mathbf{A}_o = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \qquad \mathbf{B}_o = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{C}_o = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D_o = 0$$

Observer Canonical Form

The state space representation of the following transfer function

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n}$$

in observer canonical form is given by :

$$\mathbf{A}_o = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & \cdots & 1 \\ -a_n & 0 & \cdots & \cdots & 0 \end{bmatrix} \qquad \mathbf{B}_o = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

$$\mathbf{C}_o = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \end{bmatrix} \qquad D_o = 0$$

State-Space Models

Exercise

Find a state-space representation for : $G(s) = \dfrac{K}{\tau s + 1}$.

(A)	$A = -1/\tau$	$B = K$	$C = 1$	$D = 0$
(B)	$A = -\tau$	$B = K$	$C = 1$	$D = 0$
(C)	$A = 1/\tau$	$B = K/\tau$	$C = 1$	$D = 0$
(D)	$A = -1/\tau$	$B = 1$	$C = K/\tau$	$D = 0$

Exercise

Find a state-space representation for : $G(s) = \dfrac{s+1}{s+2}$

(A)	$A = -2$	$B = 1$	$C = 1$	$D = 0$
(B)	$A = -2$	$B = -1$	$C = 1$	$D = 1$
(C)	$A = -2$	$B = 1$	$C = -1$	$D = 0$
(D)	$A = -2$	$B = 1$	$C = 1$	$D = 1$

State-Space Models

Similarity Transformation

- State-space representation of a system is not unique.
- Some representations (e.g. modal, control or observer canonical forms) can be easily obtained from the system transfer function.
- All representations are related by a state transformation. If $\mathbf{x}$ is the state vector of a system then $\mathbf{z} = \mathbf{T}\mathbf{x}$ is also a state vector of the system where $\mathbf{T} \in \mathbb{R}^{n \times n}$ is a nonsingular matrix.

$$\dot{\mathbf{z}} = \mathbf{T}\dot{\mathbf{x}} = \mathbf{TAx} + \mathbf{TB}u = \mathbf{TAT}^{-1}\mathbf{z} + \mathbf{TB}u$$
$$y = \mathbf{Cx} + Du = \mathbf{CT}^{-1}\mathbf{z} + Du$$

Therefore, if $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$ is a state-space model of a system then $(\mathbf{TAT}^{-1}, \mathbf{TB}, \mathbf{CT}^{-1}, D)$ is also a state-space model for the same system.

State-Space Models

Controllability

A state-space model is completely controllable if there is a control signal $u(t)$ that can transfer any initial state $\mathbf{x}(0)$ to any other desired location $\mathbf{x}(t)$ in a finite time. For a controllable system the controllability matrix $\mathcal{C}$ is nonsingular.

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

Observability

A state-space model is completely observable if the initial state $\mathbf{x}(0)$ can be determined from the observation history $y(t)$ in a finite time, given the control $u(t)$. For an observable system, the observability matrix $\mathcal{O}$ is nonsingular.

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

Remark : Control canonical form is always *controllable* and observer canonical form is always *observable*.

State-Space Models

Example

Is the following state-space model controllable and observable?

$$\mathbf{A} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Solution : We compute the controllability and observability matrices :

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 2 & -12 \end{bmatrix} \quad \Rightarrow \quad \det[\mathcal{C}] = -2 \neq 0$$

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -7 & 1 \end{bmatrix} \quad \Rightarrow \quad \det[\mathcal{O}] = 1 \neq 0$$

The state-space model is controllable and observable.

State-Space Models

Exercise

Consider the system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The system is :

(A) Controllable, observable

(B) Not Controllable, not observable

(C) Controllable, not observable

(D) Not Controllable, observable

State-Space Models

From SS model to TF model

Given the following state-space model, find the transfer function of the system.

$$\boxed{\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)\\ y(t) &= \mathbf{C}\mathbf{x}(t) + Du(t)\end{aligned}}$$

Taking the Laplace transform (assuming zero initial condition), we obtain :

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \quad \Rightarrow \quad (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$

Therefore, $\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$ and the output is :

$$Y(s) = \mathbf{C}\mathbf{X}(s) + DU(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) + DU(s)$$

The transfer function of the system is :

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D := \left[\begin{array}{c|c}\mathbf{A} & \mathbf{B}\\ \hline \mathbf{C} & D\end{array}\right]}$$

Example (From SS model to TF model)

Find the transfer function of the following state-space model :

$$\mathbf{A} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$

Solution :

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s+7 & -1 \\ 12 & s \end{bmatrix} \quad \Rightarrow \quad (s\mathbf{I} - \mathbf{A})^{-1} = \frac{\begin{bmatrix} s & 1 \\ -12 & s+7 \end{bmatrix}}{s(s+7)+12}$$

Since $G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$, we compute

$$G(s) = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & 1 \\ -12 & s+7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{s^2+7s+12} = \frac{\begin{bmatrix} s & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{s^2+7s+12} = \frac{s+2}{s^2+7s+12}$$

State-Space Models

Exercise

For the following state-space model

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t) \qquad y(t) = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \mathbf{x}(t)$$

Find the transfer function model $G(s)$ between $U(s)$ and $Y(s)$.

(A) $G(s) = \frac{5s^2+32s+35}{s^3+9s^2+23s+15}$ **(B)** $G(s) = \frac{5s^2+32s+35}{s^4+9s^2+23s+15}$

(C) $G(s) = \frac{2s^2+16s+22}{s^3+9s^2+23s+15}$ **(D)** $G(s) = \frac{5s+32}{s^2+32s+9}$

State-Space Models

Time- and Frequency-Domain Analysis

By converting the state-space model to a transfer function using

$$\boxed{G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D}$$

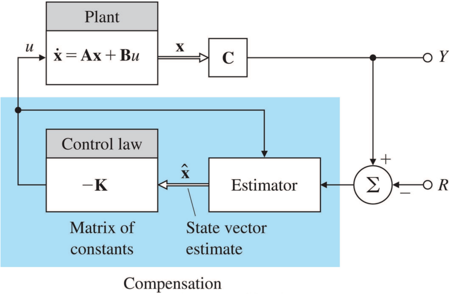
the response of the system to any input can be analysed.

Stability

- The poles of $G(s)$ are the roots of $\det(s\mathbf{I} - \mathbf{A}) = 0$.
- The eigenvalues of $\mathbf{A}$ are the roots of $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$. Therefore, the eigenvalues of $\mathbf{A}$ are the poles of $G(s)$.

A state-space model is stable if the eigenvalues of $\mathbf{A}$ are all in the LHP.

Feedback Control of State-Space Models



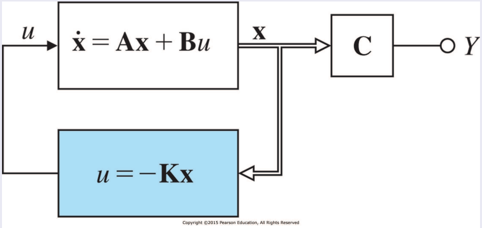
- Design a state feedback controller assuming all states are measurable.

$$u = -\mathbf{K}\mathbf{x} = -\begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Design a state estimator or *observer* .
- Combine the state estimator and the state feedback $u = -\mathbf{K}\hat{\mathbf{x}}$.

State Feedback Controller

State feedback for autonomous operation



- We assume that all states are measurable.
- We assume that the system is in *autonomous* mode, i.e., there is no reference and disturbance input.
- We should compute the feedback gain **K** such that we have *good* closed-loop dynamics. Two approaches are investigated :
  1. Pole Placement
  2. Optimal Control

State Feedback Controller

Closed-loop Pole Placement

- The closed-loop modes are related to the place of closed-loop poles.
- Good closed-loop dynamics can be achieved by assigning the desired closed-loop poles.

Closed-loop state-space model : Using a state feedback controller $u = -\mathbf{K}\mathbf{x}$ the closed-loop state space model can be computed (assuming $D = 0$) :

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \\ y &= \mathbf{C}\mathbf{x} \end{aligned}$$

- The closed-loop poles are the roots of $\det[s\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})] = 0$ or the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$.
- They can be assigned in any desired place if the system is controllable.

State Feedback Controller (Pole Placement)

Example

The state variables of a cruise control system are $\mathbf{x}(t) = \begin{bmatrix} x(t) & v(t) \end{bmatrix}$ and the state space model is :

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{-b}{m}x_2(t) + \frac{1}{m}u(t) \\ y(t) &= x_2(t) \end{aligned} \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-b}{m} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad D = 0$$

With $m = 1000$ and $b = 50$ compute the controller $\mathbf{K} = [K_1 \quad K_2]$ such that the closed-loop poles are placed at $p_1 = -0.1$ and $p_2 = -0.2$.

Solution :

$$\begin{aligned} \det(s\mathbf{I} - (\mathbf{A} - \mathbf{BK})) &= \det\left(\begin{bmatrix} s & -1 \\ 0 & s + 0.05 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix}\begin{bmatrix} K_1 & K_2 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} s & -1 \\ 0.001K_1 & s + 0.05 + 0.001K_2 \end{bmatrix}\right) \\ &= s^2 + (0.05 + 0.001K_2)s + 0.001K_1 = (s + 0.1)(s + 0.2) \end{aligned}$$

which leads to $K_1 = 20$ **and** $K_2 = 250$.

State Feedback Controller (Pole Placement)

General Solution

Assume that the desired closed-loop characteristic polynomial is

$$\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$$

Then the state feedback controller **K** is computed from the following equation :

$$\det(s\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})) = \alpha_c(s)$$

- If the state-space model is in control canonical form, the solutions becomes trivial.
- If the state-space model is controllable, there are two solutions :
  1. Compute the control canonical form representation of the system transfer function :

     $$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

     Then the solution becomes trivial.
  2. Use the Ackermann's formula.

State Feedback Controller (Pole Placement)

Trivial Solution

Consider the state-space model in the control canonical form :

$$\mathbf{A}_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \end{bmatrix} \qquad \mathbf{B}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then the closed-loop system matrix $\mathbf{A}_c - \mathbf{B}_c\mathbf{K}$ becomes :

$$\mathbf{A}_c - \mathbf{B}_c\mathbf{K} = \begin{bmatrix} -a_1 - K_1 & -a_2 - K_2 & \cdots & \cdots & -a_n - K_n \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \end{bmatrix}$$

State Feedback Controller (Pole Placement)

Trivial Solution(suit)

Therefore the closed-loop characteristic equation will be :

$$s^n + (a_1 + K_1)s^{n-1} + (a_2 + K_2)s^{n-2} + \cdots + (a_n + K_n) = 0$$

The desired closed-loop characteristic equation is :

$$\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n = 0$$

which leads to : $K_i = -a_i + \alpha_i$ for $i = 1, \ldots, n$.

Example

For the following state-space model compute a controller $\mathbf{K}$ to have $\alpha_c(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$ with $\zeta = 0.7$ and $\omega_n = 4$.

$$\mathbf{A} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad , \quad D = 0$$

Solution : we have $K_1 = -7 + 2(0.7)(4) = -1.4$ and $K_2 = -12 + 16 = 4$.

State Feedback Controller (Pole Placement)

Ackermann's formula

The controller is given by :

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \mathcal{C}^{-1} \alpha_c(\mathbf{A})$$

where $\mathcal{C}$ is the controllability matrix

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \ldots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

and $\alpha_c(\mathbf{A})$ is a matrix defined as :

$$\alpha_c(\mathbf{A}) = \mathbf{A}^n + \alpha_1 \mathbf{A}^{n-1} + \alpha_2 \mathbf{A}^{n-2} + \cdots + \alpha_n \mathbf{I}$$

The Ackermann's formula shows clearly that the pole placement problem has a solution if and only if the state-space model is controllable.

State Feedback Controller (Pole Placement)

Example

Using the Ackermann's formula compute a controller **K** for the following state-space model :

$$\mathbf{A} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad , \quad D = 0$$

to have $\alpha_c(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$ with $\zeta = 0.7$ and $\omega_n = 4$.

Solution : we have

$$\mathcal{C} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 1 & -5 \\ 2 & -12 \end{bmatrix} \quad \Rightarrow \quad \mathcal{C}^{-1} = \frac{1}{-2}\begin{bmatrix} -12 & 5 \\ -2 & 1 \end{bmatrix}$$

$$\alpha_c(\mathbf{A}) = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix}^2 + 5.6\begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} + \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix} = \begin{bmatrix} 13.8 & -1.4 \\ 16.8 & 4 \end{bmatrix}$$

$$\mathbf{K} = [0 \quad 1]\begin{bmatrix} 6 & -2.5 \\ 1 & -0.5 \end{bmatrix}\begin{bmatrix} 13.8 & -1.4 \\ 16.8 & 4 \end{bmatrix} = [5.4 \quad -3.4]$$

State Feedback Controller (Pole Placement)

Exercise

Consider the system : $G(s) = \dfrac{3s^2 + 5s - 5}{s^3 + 12s^2 + 10s + 5}$.

Determine a state feedback controller $\mathbf{K}$ so that the closed-loop poles are -3, -4 and -6.

$$\begin{array}{ll} \textbf{(A)}\mathbf{K} = [1 \quad 44 \quad 67] & \textbf{(B)}\mathbf{K} = [10 \quad 44 \quad 67] \\ \textbf{(C)}\mathbf{K} = [44 \quad 1 \quad 1] & \textbf{(D)}\mathbf{K} = [1 \quad 67 \quad 44] \end{array}$$

State Feedback Controller (Pole Placement)

How to choose the desired closed-loop poles

- For an n-th order system, we can assign n closed-loop poles.
- The closed-loop modes are $e^{p_i t}$, so smaller poles give faster response and faster dynamics.
- Faster response requires larger control signal $u(t)$. Since the magnitude of $u(t)$ is always limited, we cannot increase the speed of the response arbitrarily.
- A typical choice is a second-order polynomial as $s^2 + 2\zeta\omega_n s + \omega_n^2$ whose roots represent the dominant dynamics and some auxiliary non dominant fast poles : $\alpha_c(s) = (s^2 + 2\zeta\omega_n s + \omega_n^2)(s + \alpha)^{n-2}$, where $\zeta = 0.7$ and ω_n is close to the desired bandwidth. The fast pole α is chosen much larger than ω_n, i.e. $\alpha \gg \omega_n$.
- For stable plant models, if we choose the dominant closed-loop poles equal to the dominant poles of the plant model, the energy of the control signal is reduced.

State Feedback Controller (Optimal Control)

Linear Quadratic Regulator (LQR)

Consider a linear multivariable system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ with the controller $\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t)$. Compute $\mathbf{K}$ such that the following performance criterion is minimized :

$$\boxed{\mathcal{J} = \int_0^\infty [\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)]dt}$$

where $\mathbf{Q} = \mathbf{Q}^T \geq 0$ (positive semi-definite) and $\mathbf{R} = \mathbf{R}^T > 0$ (positive definite) matrices that determine the relative importance of the states and the control signal.

Solution : The optimal controller is $\boxed{\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}}$ when $\mathbf{P} = \mathbf{P}^T > 0$ is the solution of the following *Riccati Equation* :

$$\boxed{\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = 0}$$

State Feedback Controller (Optimal Control)

Proof of the LQR control problem

We replace $\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t)$ in the criterion :

$$\begin{aligned}\mathcal{J} &= \int_0^\infty [\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{x}^T(t)\mathbf{K}^\mathbf{T}\mathbf{R}\mathbf{K}\mathbf{x}(t)]dt \\ &= \int_0^\infty \mathbf{x}^T(t)(\mathbf{Q} + \mathbf{K}^\mathbf{T}\mathbf{R}\mathbf{K})\mathbf{x}(t)dt\end{aligned}$$

We postulate the existence of an exact differential so that :

$$\mathbf{x}^T(t)(\mathbf{Q} + \mathbf{K}^\mathbf{T}\mathbf{R}\mathbf{K})\mathbf{x}(t) = -\frac{d}{dt}[\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)]$$

where $\mathbf{P} = \mathbf{P}^T > 0$ is a matrix to be determined. Then

$$\begin{aligned}\mathbf{x}^T(t)(\mathbf{Q} + \mathbf{K}^\mathbf{T}\mathbf{R}\mathbf{K})\mathbf{x}(t) &= -\dot{\mathbf{x}}^T(t)\mathbf{P}\mathbf{x}(t) - \mathbf{x}^T(t)\mathbf{P}\dot{\mathbf{x}}(t) \\ &= -\mathbf{x}^T(t)[(\mathbf{A} - \mathbf{B}\mathbf{K})^T\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K})]\mathbf{x}(t)\end{aligned}$$

State Feedback Controller (Optimal Control)

Proof of the LQR control problem (suit)

Therefore, we require $(\mathbf{A}-\mathbf{BK})^T\mathbf{P}+\mathbf{P}(\mathbf{A}-\mathbf{BK}) = -(\mathbf{Q}+\mathbf{K^T}\mathbf{RK})$.
Rewriting the equation gives the following *Riccati* equation :

$$\mathbf{A}^T\mathbf{P}+\mathbf{PA}-\mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P}+\mathbf{Q}+(\mathbf{K}^T-\mathbf{PBR}^{-1})\mathbf{R}(\mathbf{K}-\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}) = 0$$

On the other hand the criterion can be written as :

$$\begin{aligned}\mathcal{J} &= \int_0^\infty \mathbf{x}^T(t)(\mathbf{Q}+\mathbf{K^T}\mathbf{RK})\mathbf{x}(t)dt = \int_0^\infty \frac{-d}{dt}\mathbf{x}^T(t)\mathbf{Px}(t)dt \\ &= -\mathbf{x}^T(t)\mathbf{Px}(t)\Big|_0^\infty = -\mathbf{x}^T(\infty)\mathbf{Px}(\infty)+\mathbf{x}^T(0)\mathbf{Px}(0)\end{aligned}$$

For stable systems, the first term is zero. The smallest $\mathbf{P}$ (that minimizes $\mathcal{J}$) is obtained when the last term of the Riccati equation is zero, i.e.

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \quad \text{and} \quad \mathbf{A}^T\mathbf{P}+\mathbf{PA}-\mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P}+\mathbf{Q} = 0$$

This can be proved using *the comparison Lemma*.

State Feedback Controller (Optimal Control)

Comparison Lemma

Consider $\mathbf{P}_1$ and $\mathbf{P}_2$ as positive definite solutions to the Riccati equations :

$$\mathbf{A}^T\mathbf{P}_1 + \mathbf{P}_1\mathbf{A} - \mathbf{P}_1\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_1 + \mathbf{Q}_1 = 0$$
$$\mathbf{A}^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{A} - \mathbf{P}_2\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_2 + \mathbf{Q}_2 = 0$$

If $\mathbf{Q}_1 \leq \mathbf{Q}_2$, then $\mathbf{P}_1 \leq \mathbf{P}_2$.

Proof of the LQR control problem (suit)

The optimal controller minimizes $\mathcal{J} = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0)$ for all initial conditions, where $\mathbf{P} > 0$ is a solution to :

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} + (\mathbf{K}^T - \mathbf{P}\mathbf{B}\mathbf{R}^{-1})\mathbf{R}(\mathbf{K} - \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}) = 0$$

Take $\mathbf{Q}_1 = \mathbf{Q}$ and $\mathbf{Q}_2 = \mathbf{Q} + (\mathbf{K}^T - \mathbf{P}\mathbf{B}\mathbf{R}^{-1})\mathbf{R}(\mathbf{K} - \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P})$.

Since the second term is always positive we have $\mathbf{Q}_1 \leq \mathbf{Q}_2$. Therefore, according to the comparison lemma, the smallest $\mathbf{P}$ is obtained when the second term is zero, which leads to the optimal controller : $\boxed{\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}}$

State Feedback Controller (Optimal Control)

How to design an LQR controller

A Matlab command can be used as : `K=lqr(A,B,Q,R);`

- We can start with $\mathbf{Q} = \mathbf{I}$ and simulate the states of the system. We can then increase (or decrease) the diagonal values of $\mathbf{Q}$ to have faster (or slower) convergence of the corresponding states.
- We can start with $\mathbf{R} = \mathbf{I}$ and simulate the inputs of the system. We can then increase (or decrease) the diagonal values of $\mathbf{R}$ to decrease (or increase) the magnitude of the control signals.
- We can minimize the output of the system (instead of the states) : Note that $\mathbf{y} = \mathbf{C}\mathbf{x}$ so $\mathbf{y}^T\mathbf{y} = \mathbf{x}^T\mathbf{C}^T\mathbf{C}\mathbf{x}$, which leads to $\mathbf{Q} = \mathbf{C}^T\mathbf{C}$.
- In general, $\mathbf{Q}$ and $\mathbf{R}$ are tuned iteratively. A good starting value is diagonal matrices with

$$Q_{ii} = 1/\text{maximum acceptable value of } x_i^2$$
$$R_{ii} = 1/\text{maximum acceptable value of } u_i^2$$

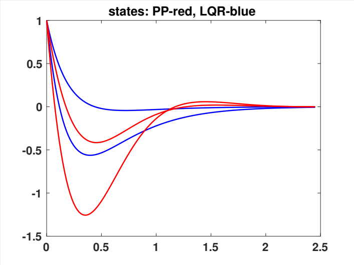
State Feedback Controller (Optimal Control)

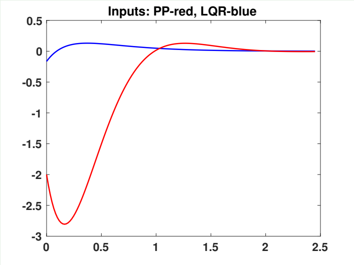
Example (LQR versus Pole Placement)

Consider the following system :

$$\mathbf{A} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad , \quad D = 0$$

Pole placement : Design a controller to have $\alpha_c(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$ with $\zeta = 0.7$ and $\omega_n = 4$.
LQR : Design a controller with $\mathbf{Q} = \mathbf{I}$ and $R = 1$.





State Feedback Controller (Optimal Control)

Example (LQR versus Pole Placement)

```
% Simulation model
A=[-7 1;-12 0];B=[1;2];C=[1,0];D=0;

% Pole placement
zeta=0.7; wn=4; Pd=roots([1 2*zeta*wn wn^2]);

K_pp=acker(A,B,Pd)

% Closed-loop simulation
CL_pp=ss(A-B*K_pp,[],C,[]);
x0=[1;1];

[y_pp,t,x_pp]=initial(CL_pp,x0);
u_pp=-K_pp*x_pp';

% LQR design
R=1;
K_lqr=lqr(A,B,eye(2),R);

% Closed-loop simulation
CL_lqr=ss(A-B*K_lqr,[],C,[])
[y_lqr,t,x_lqr]=initial(CL_lqr,x0,t);
u_lqr=-K_lqr*x_lqr';
```

State Feedback Controller

Exercise

Consider a first order unstable system : $G(s) = \dfrac{3}{s-2}$

1. Give a state-space representation of the system.

   (A) $A = -2, B = 3, C = 1$ **(B)** $A = 2, B = 1, C = 3$
   (C) $A = 2, B = 1, C = -3$ **(D)** $A = -2, B = -1, C = 3$

2. Compute an LQR controller that minimizes $\mathcal{J} = \int_0^\infty [y^2(t) + u^2(t)]dt$.

   (A) $K = 2 \pm \sqrt{13}$ **(B)** $K = 2 - \sqrt{13}$
   (C) $K = 2 + \sqrt{13}$ **(D)** None of the above

State Feedback Controller

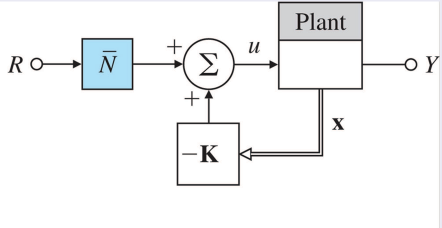
Introducing the reference input

A reference input $r(t)$ with a feedforward gain can be added to the control signal as follows :

$$u(t) = -\mathbf{K}\mathbf{x}(t) + \bar{N}r(t)$$

Therefore, the state-space equation of the closed-loop system becomes :

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}(-\mathbf{K}\mathbf{x}(t) + \bar{N}r(t)) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\bar{N}r(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$



State Feedback Controller

Compute $\bar{N}$ to have zero steady-state tracking error

- Assume that the closed-loop system is stable and the reference signal $r(t)$ converges to a constant value r_{ss} at steady-state.
- The objective is to find $\bar{N}$ such that at steady-state $y(\infty) = r_{ss}$.
- At steady-state we have

$$\dot{\mathbf{x}}(\infty) = 0 \qquad \mathbf{x}(\infty) = \mathbf{x}_{ss} \qquad y(\infty) = y_{ss} \qquad r(\infty) = r_{ss}$$

- This leads to the following equations at steady-state :

$$\begin{aligned} \mathbf{0} &= (\mathbf{A} - \mathbf{BK})\mathbf{x}_{ss} + \mathbf{B}\bar{N}r_{ss} \\ y_{ss} &= \mathbf{C}\mathbf{x}_{ss} \end{aligned}$$

- Therefore,

$$y_{ss} = \mathbf{C}(-\mathbf{A} + \mathbf{BK})^{-1}\mathbf{B}\bar{N}r_{ss} \quad \Rightarrow \quad \boxed{\bar{N} = \left[\mathbf{C}(-\mathbf{A} + \mathbf{BK})^{-1}\mathbf{B}\right]^{-1}}$$

State Feedback Controller

Example (Introducing the reference input)

For the following system compute the feedforward gain for zero steady-state error of a step reference.

$$\mathbf{K} = [5.4 \quad -3.4] \quad \mathbf{A} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0$$

Solution :

$$\begin{aligned}
\mathbf{C}(-\mathbf{A} + \mathbf{BK})^{-1}\mathbf{B} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 7 & -1 \\ 12 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} [5.4 \quad -3.4] \right)^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -0.425 & 0.275 \\ -1.425 & 0.775 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= 0.125 \quad \Rightarrow \quad 0.125\bar{N} = 1 \quad \Rightarrow \quad \bar{N} = 8
\end{aligned}$$

State Feedback Controller (Integral Control)

- Using a feedforward gain $\bar{N}$ the steady-state tracking error for a step signal goes to zero.
- $\bar{N}$ depends on the model parameters. Therefore, in the presence of modelling error, the tracking error will not converge to zero.

Solution : Add an integrator into the loop !

- Define a new artificial state

$$x_I(t) = \int_0^t e(\tau)d\tau \qquad \text{where} \quad e(t) = r(t) - y(t) \quad \text{and} \quad \dot{x}_I(t) = e(t)$$

  Therefore, we have a new state equation : $\dot{x}_I(t) = -\mathbf{C}\mathbf{x} + r(t)$
- Make an augmented model with the integrator :

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_I \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r$$

- Design a state feedback controller $\mathbf{K}$ for the augmented plant.
- At steady-state $\dot{x}_I(t) = e(t) = 0$.

State Feedback Controller (Integral Control)

Example (State feedback with integral action)

For the following system compute a state feedback controller for zero steady-state error of a step reference.

$$\mathbf{A} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0$$

Solution : We compute first the augmented plant matrices

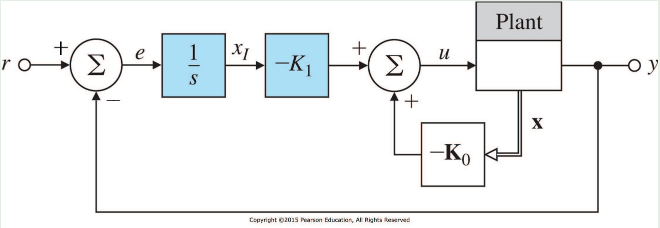
$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} = \begin{bmatrix} -7 & 1 & 0 \\ -12 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

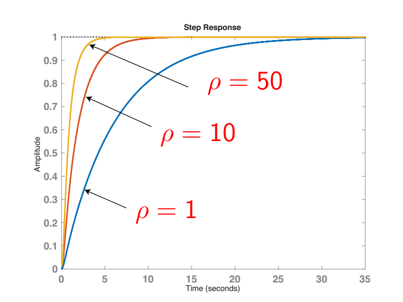
Then we can design a state feedback controller $\mathbf{K} = [\mathbf{K}_0 \quad K_1]$ for the augmented plant using pole placement or LQR method.

```
Abar=[A zeros(2,1);-C 0];Bbar=[B;0];
Q=rho*eye(3);R=1;
K=lqr(Abar,Bbar,Q,R);
```

State Feedback Controller (Integral Control)

Example (State feedback with integral action)





Estimator or Observer Design

- The full state feedback controller is given by $u = -\mathbf{K}\mathbf{x} + \bar{N}r$.
- In practice the states are not all measurable. If an estimate of the states $\hat{\mathbf{x}}$ is available we can compute the control signal as : $u = -\mathbf{K}\hat{\mathbf{x}} + \bar{N}r$.
- We can estimate the states if we know the model of the plant :

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u$$

  However, we need $\hat{\mathbf{x}}(0)$ which is usually unknown.
- The dynamics of the estimation error $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$ is given by

$$\dot{\tilde{\mathbf{x}}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}}, \qquad \tilde{\mathbf{x}}(0) = \mathbf{x}(0) - \hat{\mathbf{x}}(0)$$

- The estimation error has the same dynamics as the plant model.
  - If the plant model is unstable, the estimation error diverges.
  - We need stable and fast dynamics for state estimation.

Solution : Use feedback to stabilize the estimator and improve its dynamics. *The dynamics of the estimator should be 3 to 10 (or 2 to 6 in your reference book) times faster than the control dynamics.*

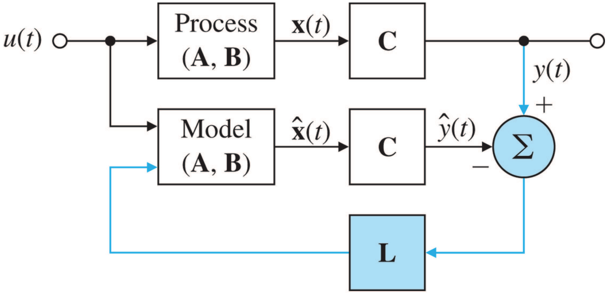
Estimator or Observer Design

Full-Order Estimator Design : Define the estimator equation as

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}})$$

where the estimator gain $\mathbf{L} = [l_1, l_2, \ldots, l_n]^T$ is designed to have satisfactory error dynamics :

$$\dot{\tilde{\mathbf{x}}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{LC})\tilde{\mathbf{x}}$$



Estimator or Observer Design

Estimator Design by Pole Placement : Define the desired estimator characteristic polynomial as $\alpha_e(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$.
Then, compute $\mathbf{L}$ such that $\det(s\mathbf{I} - \mathbf{A} + \mathbf{LC}) = \alpha_e(s)$.
Trivial Solution : Suppose that the plant model is given in observer canonical form :

$$\mathbf{A}_o = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & & 1 \\ -a_n & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \mathbf{B}_o = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad \begin{matrix} \mathbf{C}_o = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \\ D_o = 0 \end{matrix}$$

Then the closed-loop state matrix for the estimator error is :

$$\mathbf{A}_o - \mathbf{LC}_o = \begin{bmatrix} -a_1 - l_1 & 1 & 0 & \cdots & 0 \\ -a_2 - l_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & & 1 \\ -a_n - l_n & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

and the characteristic equation is $s^n + (a_1 + l_1)s^{n-1} + \cdots + (a_n + l_n) = 0$ which leads to $l_i = \alpha_i - a_i$ for $i = 1, \ldots, n$.

Estimator or Observer Design

Ackermann's Formula

The Estimator gain is given by :

$$\mathbf{L} = \alpha_e(\mathbf{A})\mathcal{O}^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \alpha_e(\mathbf{A})\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

where $\mathcal{O}$ is the observability matrix and $\alpha_e(\mathbf{A})$ is a matrix defined as :

$$\alpha_e(\mathbf{A}) = \mathbf{A}^n + \alpha_1\mathbf{A}^{n-1} + \cdots + \alpha_n\mathbf{I}$$

The Ackermann's formula shows clearly that the estimator design has a solution if and only if the state-space model is observable.

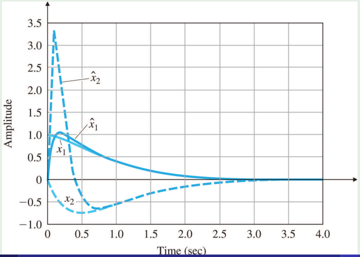
Example (Estimator Design by Pole Placement)

The state-space model of a system is given by :

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0$$

Design $\mathbf{L} = [l_1 \quad l_2]^T$ to place the two estimator error poles at -10.

Solution : The estimator characteristic polynomial is $\alpha_e(s) = (s+10)^2 = s^2 + 20s + 100$. Since the state-space model is in observer canonical form, $l_1 = 20 - 0 = 20$ and $l_2 = 100 - 1 = 99$.



Estimator or Observer Design

Design by Duality

- State feedback design and estimator design are mathematically equivalent. This property is called **duality**.
- Note that $\mathbf{A}-\mathbf{LC}$ and $\mathbf{A}^T-\mathbf{C}^T\mathbf{L}^T$ have the same eigenvalues. So computing $\mathbf{L}^T$ to place the eigenvalues of $\mathbf{A}^T-\mathbf{C}^T\mathbf{L}^T$ is the same problem as computing $\mathbf{K}$ to place the eigenvalues of $\mathbf{A}-\mathbf{BK}$.
- We can make a duality table :

Control	Estimation
$\mathbf{A}$	$\mathbf{A}^T$
$\mathbf{B}$	$\mathbf{C}^T$
$\mathbf{C}$	$\mathbf{B}^T$

- The Matlab command `acker` and `lqr` can be used for controller design as well as for the estimator design.

`K=acker(A,B,Pc);`	`K=lqr(A,B,Q,R)`
`Lt=acker(A',C',Pe); L=Lt';`	`Lt=lqr(A',C',Q,R);L=Lt';`

Estimator or Observer Design

Exercise

Given the transfer function of a system : $G(s) = \dfrac{2}{0.2s - 1}$

1. Find a state-space model for the system.
   (A) $A = 0.2, B = 1, C = 2$ **(B)** $A = 5, B = 1, C = 10$
   (C) $A = -5, B = 10, C = 1$ **(D)** $A = -0.2, B = 2, C = 1$
2. Compute the gain of observer L as a function of Q (take $R = 1$).
   (A) $L = 5 + \sqrt{100 + Q}$ **(B)** $L = 5 - \sqrt{100 + Q}$
   (C) $L = 0.5 + \sqrt{0.25 + Q}$ **(D)** $L = 0.5 - \sqrt{0.25 + Q}$

Combining Control and Estimator Design

- The control signal in the autonomous case is $u = -\mathbf{K}\hat{\mathbf{x}}$
- The plant equation with feedback is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}(\mathbf{x} - \tilde{\mathbf{x}})$$

- The estimator error equation is $\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\tilde{\mathbf{x}}$.
- Combining these equations, we obtain :

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\tilde{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \tilde{\mathbf{x}} \end{bmatrix}$$

- The characteristic equation of the closed-loop system is :

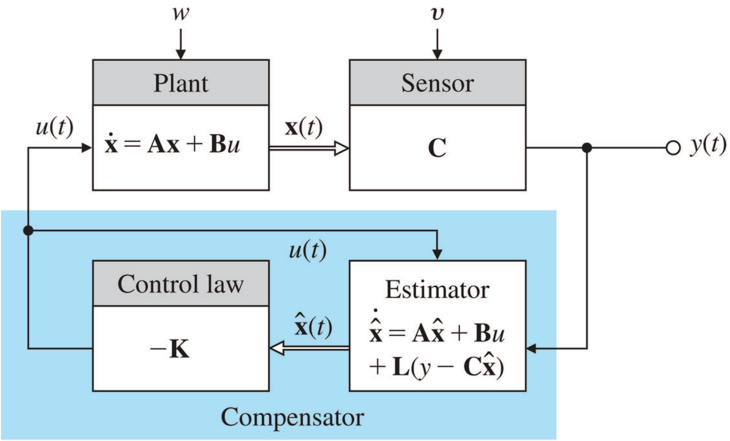
$$\begin{aligned} \det \begin{bmatrix} s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K} & -\mathbf{B}\mathbf{K} \\ \mathbf{0} & s\mathbf{I} - \mathbf{A} + \mathbf{L}\mathbf{C} \end{bmatrix} &= \det(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K})\det(s\mathbf{I} - \mathbf{A} + \mathbf{L}\mathbf{C}) \\ &= \alpha_c(s)\alpha_e(s) = 0 \end{aligned}$$

- The closed-loop poles are the poles of controller and the poles of estimator.

Separation Principle

The designs of control law and estimator can be done independently.

Combining Control and Estimator Design



$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{L}y \\ u &= -\mathbf{K}\hat{\mathbf{x}} \end{aligned} \quad \Rightarrow \quad D_c(s) = \frac{U(s)}{Y(s)} = -\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K} + \mathbf{L}\mathbf{C})^{-1}\mathbf{L}$$

Combining Control and Estimator Design

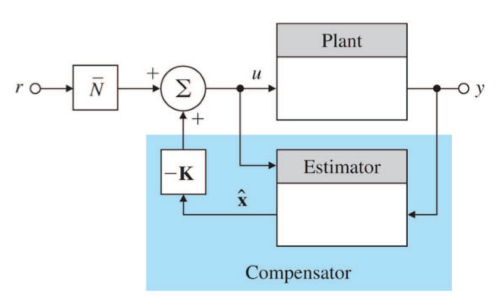
Exercise

Design a compensator for $G(s) = 1/s^2$ such that the control poles are at $\omega_n = 1$ rad/s and $\zeta = 0.707$ and the estimator poles all at -5.

1. Give an observer canonical representation for the system.
   - **(A)** $A = [0 \quad 1; 1 \quad 0]$ $B = [1 \quad 0]^T$ $C = [1 \quad 0]$
   - **(B)** $A = [0 \quad 1; 0 \quad 0]$ $B = [0 \quad 1]^T$ $C = [1 \quad 0]$
   - **(C)** $A = [0 \quad 1; 0 \quad 0]$ $B = [0 \quad 1]^T$ $C = [1 \quad 1]$
   - **(D)** $A = [0 \quad 0; 1 \quad 0]$ $B = [1 \quad 1]^T$ $C = [1 \quad 0]$
2. Compute the controller $\mathbf{K}$:
   - **(A)** $\mathbf{K} = [1 \quad 0.707]$ **(B)** $\mathbf{K} = [1 \quad \sqrt{2}]$
   - **(C)** $\mathbf{K} = [\sqrt{2} \quad 1]$ **(D)** $\mathbf{K} = [0.707 \quad \sqrt{2}]$
3. Compute the observer gain $\mathbf{L}$:
   - **(A)** $\mathbf{L} = [5 \quad 5]$ **(B)** $\mathbf{L} = [-5 \quad -5]$
   - **(C)** $\mathbf{L} = [25 \quad 5]$ **(D)** $\mathbf{L} = [10 \quad 25]$

Combining Control and Estimator Design

Introduction of the reference signal



Main equations :

$$
\begin{aligned}
&\text{States of the plant :} && \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\
&\text{Output equation :} && y = \mathbf{C}\mathbf{x} \\
&\text{Estimator equation :} && \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}}) \\
&\text{Control law :} && u = -\mathbf{K}\hat{\mathbf{x}} + \bar{N}r
\end{aligned}
$$

Combining Control and Estimator Design

Introduction of the reference signal

Closed-loop equations :

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(-\mathbf{K}\hat{\mathbf{x}} + \bar{N}r) = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} + \mathbf{B}\bar{N}r$$

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}(-\mathbf{K}\hat{\mathbf{x}} + \bar{N}r) + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}}) = \mathbf{L}\mathbf{C}\mathbf{x} + (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{B}\bar{N}r$$

$$y = \mathbf{C}\mathbf{x}$$

Closed-loop state-space model :

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{C} & \mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C} \end{bmatrix}}_{\mathbf{A}_{cl}} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{B}\bar{N} \\ \mathbf{B}\bar{N} \end{bmatrix}}_{\mathbf{B}_{cl}} r \quad ; \quad y = \underbrace{[\mathbf{C} \quad \mathbf{0}]}_{\mathbf{C}_{cl}} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

Transfer function between r and y : $T(s) = \mathbf{C}_{cl}(s\mathbf{I} - \mathbf{A}_{cl})^{-1}\mathbf{B}_{cl}$

Find the transfer function between r and u : The main equations remains the same, but the output equation will be $u = -\mathbf{K}\hat{\mathbf{x}} + \bar{N}r$. Therefore :

$$U(s) = [\mathbf{0} \quad -\mathbf{K}](s\mathbf{I} - \mathbf{A}_{cl})^{-1}\mathbf{B}_{cl} + \bar{N}$$

Combining Control and Estimator Design

Computing feedforward gain

$\bar{N}$ can be computed by imposing the steady-state gain of the closed-loop transfer function (between r and y) equal to 1 :

$$\lim_{s\to 0} T(s) = \mathbf{C}_{cl}(s\mathbf{I} - \mathbf{A}_{cl})^{-1}\mathbf{B}_{cl} = 1 \quad \Rightarrow \quad \bar{N} = -\left(\mathbf{C}_{cl}\mathbf{A}_{cl}^{-1}\begin{bmatrix} \mathbf{B} \\ \mathbf{B} \end{bmatrix}\right)^{-1}$$

Introduction of the output disturbance

Suppose that the disturbance w is added to the output (assuming $r = 0$). Then, the closed-loop equations are :

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}$$

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}}) = \mathbf{L}\mathbf{C}\mathbf{x} + (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{L}w$$

$$y = \mathbf{C}\mathbf{x} + w$$

The transfer function between w and y : $\quad S(s) = \mathbf{C}_{cl}(s\mathbf{I} - \mathbf{A}_{cl})^{-1}\begin{bmatrix} \mathbf{0} \\ \mathbf{L} \end{bmatrix} + 1$

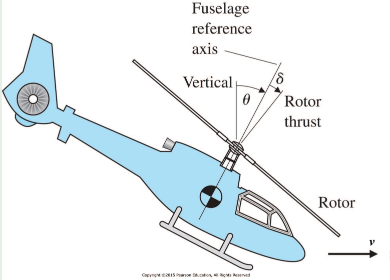
Combining Control and Estimator Design

Example

The linearized longitudinal motion of a helicopter near hover can be modeled by :

$$\begin{bmatrix} \dot{q} \\ \dot{\theta} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -0.4 & 0 & -0.01 \\ 1 & 0 & 0 \\ -1.4 & 9.8 & -0.02 \end{bmatrix} \begin{bmatrix} q \\ \theta \\ v \end{bmatrix} + \begin{bmatrix} 6.3 \\ 0 \\ 9.8 \end{bmatrix} \delta$$

Suppose that we measure the horizontal velocity v as the output, that is $y = v$.



Combining Control and Estimator Design

Example

- Find the poles of the plant model.
  Answer : Use `eig(A)` to obtain the poles as -0.6565 and $0.1183 \pm j0.3678$. Note that the plant has RHP poles and is unstable.
- Is the system controllable ?
  Answer : We compute the controllability matrix $\mathcal{C} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}]$. Then, we compute its determinant as $\det(\mathcal{C}) = 2451.3 > 0$. The system is controllable.
- Find the feedback gain that places the closed-loop poles at $s = -1 \pm j$ and $s = -2$.
  Answer : We compute the desired characteristic polynomial as : $\alpha_c(s) = (s+2)(s^2+2s+2) = s^3 + 4s^2 + 6s + 4$. Then

$$\mathbf{K} = [0 \quad 0 \quad 1]\mathcal{C}^{-1}\alpha_c(\mathbf{A}) = [0.4706 \quad 1.0 \quad 0.0627]$$

  We can use the command : `K=acker(A,B,[-2 -1+i -1-i]);`

Combining Control and Estimator Design

Example

- Design an estimator and place its poles at -8 and -4$\pm 4\sqrt{3}j$.
  Answer : We use the Ackermann's formula for the dual case : `Lt=acker(A',C',[-8 -4+4*sqrt(3)*i -4-4*sqrt(3)*i])`, which leads to $\mathbf{L} = [44.7097 \quad 18.8130 \quad 15.5800]^T$

- Compute the compensator transfer function (for a negative feedback).
  Answer : Use $D_c(s) = \mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{BK} + \mathbf{LC})^{-1}\mathbf{L}$ to compute the controller. Or use the commands `Dc=ss(A-B*K-L*C,L,K,0);tf(Dc)` :

$$D_c(s) = \frac{40.83s^2 + 60.99s + 31.88}{s^3 + 19.58s^2 - 210.4s + 814.7}$$

- Draw the Bode plot of the open-loop transfer function and indicate the gain and phase margins.
  Answer : The transfer function of the plant is `tf(ss(A,B,C,0))`

$$G(s) = \frac{9.8s^2 - 4.9s + 61.74}{s^3 + 0.42s^2 - 0.006s + 0.098}$$

Combining Control and Estimator Design

Example