Control Systems I

Quick Review of Systèmes Dynamique

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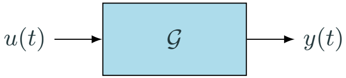
System Definition

A dynamic system transforms an input signal $u(t)$ into an output signal $y(t)$.

$$y = \mathcal{G}(u)$$



Important system properties that we will assume throughout the course:

- Linear
- Causal
- Time invariant

A system with these properties is referred to as an ***LTI system***.

System Property: Linearity

Linear System

The system $\mathcal{G}$ is called ***linear*** if for any two signals u_1, u_2 and for all numbers $a \in \mathbb{R}$

$$\mathcal{G}\left(u_1 + u_2\right) = \mathcal{G}(u_1) + \mathcal{G}(u_2)$$
$$\mathcal{G}\left(a u_1\right) = a\mathcal{G}(u_1)$$

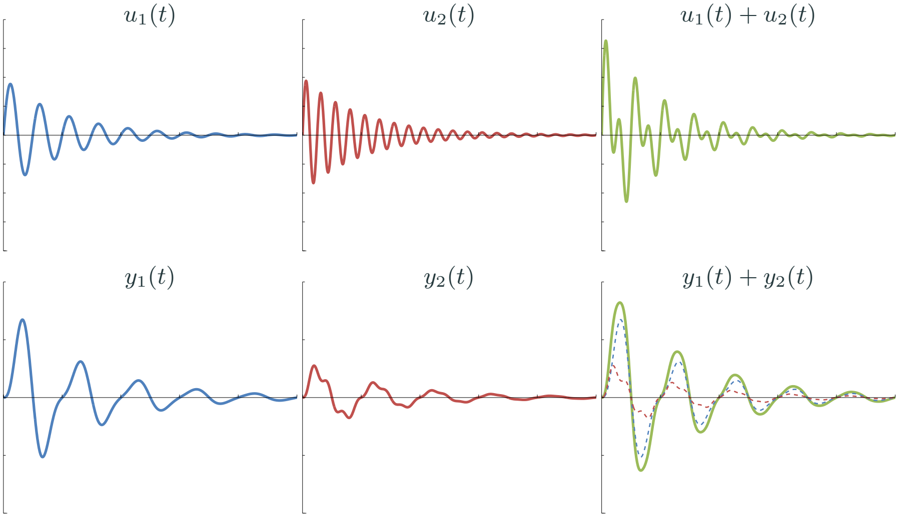
Also called ***superposition***.

A simple idea:

- If $y_i(t)$ is the output from the input signal $u_i(t)$
- then $y(t) = \sum_{i=0}^{n} a_i y_i(t)$ is the output from the signal $\sum_{i=0}^{n} a_i u_i(t)$

This is the basis of Harmonic analysis of dynamic systems.

Superposition



Causality

Causal System

A system $\mathcal{G}$ is ***causal*** if the output $y(t_0)$ at time t_0 is only a function of $\{u(t) \mid t \leq t_0\}$.

In words: The output of a system cannot depend on the future inputs

All physical systems are causal

Time Invariance

Time Invariant System

A system $\mathcal{G}$ is ***time invariant*** if for any input signal $u(t)$ and its corresponding output signal $y(t)$, then the output in response to the input signal $u(t+T)$ will be $y(t+T)$.

Same response from the system given the same input, no matter when that input is applied.

Impulse Response

Representing the System Response

Dirac Delta Function

Let $\delta(t)$ be the Dirac, or impulse function:

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad\qquad \delta(t) = 0 \text{ for all } t \neq 0$$

Impulse Response

The impulse response $g(t)$ is defined as the output of the system in response to a dirac delta function at time $t = 0$:

$$g(t) := \mathcal{G}(\delta(t))$$

If $\mathcal{G}$ is an LTI system, then the impulse response completely characterizes it.

Note that causality implies that $g(t) = 0$ for $t < 0$, because $\delta(t) = 0$ for $t < 0$

System Response

Theorem : Response of an LTI System

The output of an LTI system in response to an input signal $u(t)$ is

$$\mathcal{G}(u) = g * u$$

where $y = g * u$ if

$$y(t) = \int_0^t u(\tau) g(t-\tau) d\tau$$

Response of and LTI System - Proof

Demonstrate that the output of an LTI system is

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$$u(t) = \int_{\tau=0}^{\infty} u(\tau)\delta(t-\tau)d\tau$$

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Superposition then gives us:

$$y(t) = \int_{\tau=0}^{\infty} u(\tau)g(t-\tau)d\tau$$

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Superposition then gives us:

$$y(t) = \int_{\tau=0}^{\infty} u(\tau)g(t-\tau)d\tau$$

and causality gives

$$y(t) = \int_{\tau=0}^{t} u(\tau)g(t-\tau)d\tau$$

System Characterization: Impulse Response

Impulse response is enough to compute the output for any input signal

- The impulse response completely characterizes the system $\mathcal{G}$ by describing how it will operate on an input signal u
- This is an "input-output", or "behavioural" characterization of the system

Key limitation: Most systems have an infinitely-long impulse response

Need a more compact representation.

Transfer Functions

Laplace Transform

If $f(t)$ is a piecewise continuous function, then the Laplace transform of $f(t)$ is denoted $\mathcal{L}\left(f(t)\right)$ and is defined as

$$\mathcal{L}\left(f(t)\right) = F(s) = \int_0^\infty e^{-st} f(t) dt$$

where $s \in \mathbb{C}$ is complex

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Transfer Function

The ***transfer function*** of a system is the Laplace transform of its impulse response.

$$\mathcal{L}\{g(t)\} = G(s)$$

For LTI systems $G(s)$ is a rational polynomial function

The point: Convolution becomes multiplication

$$y = g * u \qquad \Leftrightarrow \qquad Y(s) = G(s)U(s)$$

Key Property of Laplace Transforms: Convolution = Multiplication

Show: $$\mathcal{L}\left(\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right) = F_1(s)F_2(s)$$

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Show: $$\mathcal{L}\left(\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right) = F_1(s)F_2(s)$$

$$\mathcal{L}\left\{\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right\} = \int_0^\infty e^{-st}\left(\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right)dt$$

Change the order of integration:

$$= \int_0^\infty \left(f_1(\tau)\int_\tau^\infty e^{-st}f_2(t-\tau)\,dt\right)d\tau$$

Substitute $z = t - \tau$:

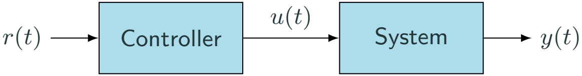
$$\int_\tau^\infty e^{-st}f_2(t-\tau)\,dt = \int_0^\infty e^{-(z+\tau)s}f_2(z)\,dz = e^{-\tau s}\int_0^\infty e^{-sz}f_2(z)\,dz = e^{-\tau s}F_2(s)$$

and therefore

$$\mathcal{L}\left\{\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right\} = \int_0^\infty f_1(\tau)e^{-\tau s}F_2(s)\,d\tau \;=\; F_2(s)\int_0^\infty f_1(\tau)e^{-\tau s}\,d\tau$$

$$= F_1(s)F_2(s)$$

Manipulation of Simple Block Diagrams



Suppose that the controller is:

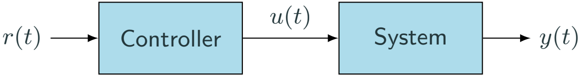
$$\ddot{u}(t) + 3u(t) = \dot{r}(t) + 4r(t)$$

and the system is

$$\ddot{y}(t) + \dot{y}(t) = u(t) + 2\dot{u}(t)$$

If we're given the reference function $r(t)$, what is $y(t)$?

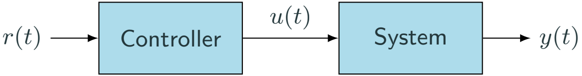
Manipulation of Simple Block Diagrams



$$\ddot{u}(t) + 3u(t) = \dot{r}(t) + 4r(t) \quad \Rightarrow \quad s^2U(s) + 3U(s) = sR(s) + 4R(s)$$

$$\ddot{y}(t) + \dot{y}(t) = u(t) + 2\dot{u}(t) \quad \Rightarrow \quad s^2Y(s) + sY(s) = U(s) + 2sU(s)$$

Manipulation of Simple Block Diagrams



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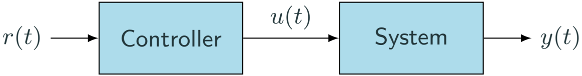
Re-arranging gives:

$$U(s) = \frac{s+4}{s^2+3}R(s) \qquad\qquad Y(s) = \frac{1+2s}{s^2+s}U(s)$$

Manipulation of Simple Block Diagrams



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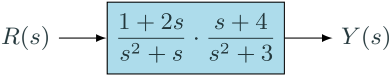
... and we can compute the impact of $r(t)$ on $y(t)$

$$Y(s) = \frac{1+2s}{s^2+s} \cdot \frac{s+4}{s^2+3}R(s)$$

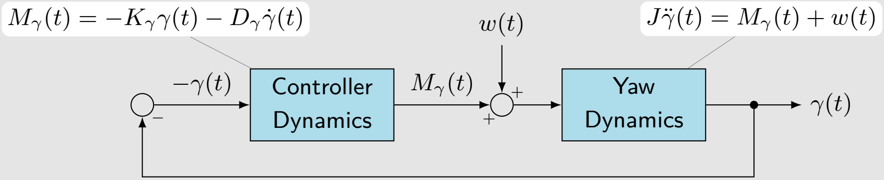
Series connection of blocks (convolution) becomes multiplication!



Closed-Loop Transfer Functions

Example: System Response

Compute response to a impulsive disturbance acting on the yaw system



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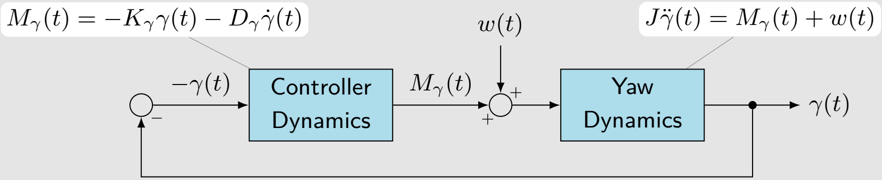

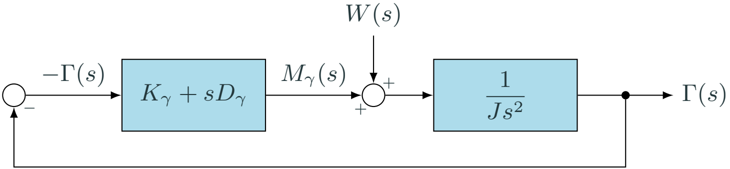

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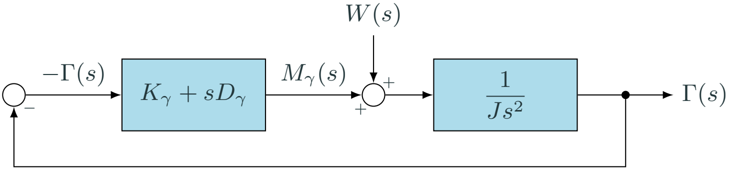

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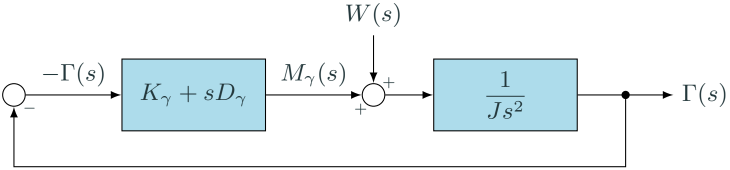
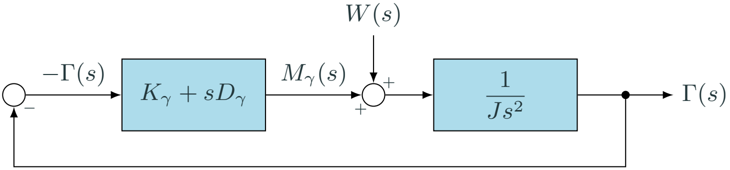


Start at the output and work backwards against the arrows

$$\Gamma = \frac{1}{Js^2}(W - (K_\gamma + sD_\gamma)\Gamma)$$

$$(Js^2 + sD_\gamma + K_\gamma)\Gamma = W$$

Example: System Response



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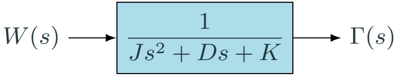
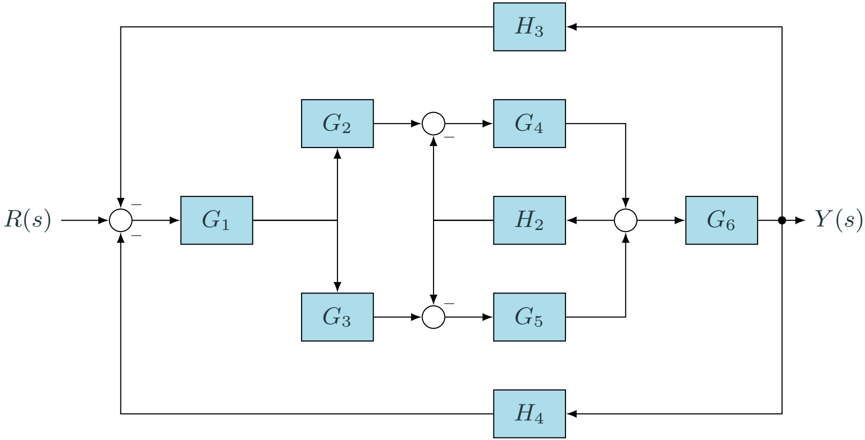


Where we recall that D sets the damping and K the response rate.

Example: Complex System



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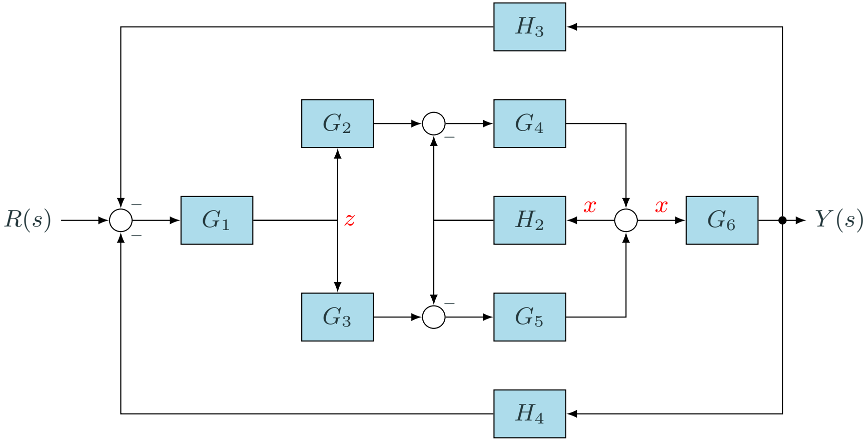

Add auxiliary variables for internal loops, and wherever convenient to simplify.

Example: Complex System

Start at the output and work back ***against*** the arrows.
A block is a multiplication, a summation is addition.

$$
\begin{aligned}
Y &= G_6 x \\
x &= G_4(G_2 z - H_2 x) + G_5(G_3 z - H_2 x) \\
z &= G_1(R - H_3 Y - H_4 Y)
\end{aligned}
$$

Solve for Y as a function of R

$$
\begin{aligned}
x &= (G_4 G_2 + G_5 G_3) z - (G_4 H_2 + G_5 H_2) x \\
(1 + G_4 H_2 + G_5 H_2) x &= (G_4 G_2 + G_5 G_3) z \\
x &= \frac{G_4 G_2 + G_5 G_3}{1 + G_4 H_2 + G_5 H_2} z \\
Y &= G_6 \frac{G_4 G_2 + G_5 G_3}{1 + G_4 H_2 + G_5 H_2} z
\end{aligned}
$$

Example: Complex System

$$Y = \underbrace{G_6 \frac{G_4 G_2 + G_5 G_3}{1 + G_4 H_2 + G_5 H_2}}_{Q} z \qquad z = G_1 R - (H_3 + H_4) Y$$

Solve to get the transfer function

$$\frac{Y}{R} = \frac{Q G_1}{1 + Q(H_3 + H_4)}$$

If we want to do more algebra, we can eliminate Q

$$\frac{Y}{R} = \frac{G_1 G_2 G_4 G_6 + G_1 G_3 G_5 G_6}{(G_4 + G_5) H_2 + G_2 G_4 G_6 H_3 + G_2 G_4 G_6 H_4 + G_3 G_5 G_6 H_3 + G_3 G_5 G_6 H_4 + 1}$$

System Response

System Response

We can now reduce our system to something of the form

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

Given any input $u(t)$, what is the value of $y(t)$?

1. Compute Laplace transform of the input $U(s) = \mathcal{L}\left[u(t)\right]$
2. Laplace transform of output is $Y(s) = G(s)U(s)$
3. Compute the inverse Laplace transform to get the output $y(t) = \mathcal{L}^{-1}\left[Y(s)\right]$

Inverse Laplace Transforms

What is the inverse Laplace transform of $Y(s) = G(s)U(s)$?

We can always write a rational polynomial $Y(s)$ as:

$$Y(s) = \frac{\prod_{i=0}^{n}(s - z_i)}{\prod_{i=0}^{m}(s - p_i)}$$

where $\{z_i\}$ are the ***zeros*** of the system, and $\{p_i\}$ are the ***poles***.

A partial fraction expansion gives:

$$Y(s) = \sum_{i=0}^{m} \frac{c_i}{s - p_i}$$

from which we can compute the inverse Laplace transform

$$y(t) = \sum_{i=0}^{m} c_i e^{p_i t}$$

we can see that the signal $y(t)$ is bounded if and only if ***all poles are in the left half plane***.

Time Domain Response

What is the yaw in response to an impulsive disturbance for $K = 24$, $D = 10$, $J = 1$

$$\frac{\Gamma}{W} = \frac{1}{Js^2 + Ds + K} = \frac{1}{s^2 + 10s + 24}$$

Impulsive input $W(s) = \mathcal{L}\left[\delta(t)\right] = 1$

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Impulsive input $W(s) = \mathcal{L}\left[\delta(t)\right] = 1$

Partial fraction expansion

$$\begin{aligned}\Gamma &= \frac{1}{s^2 + 10s + 24} W(s) = \frac{1}{s^2 + 10s + 24} \\ &= \frac{1}{(s+4)(s+6)} \\ &= \frac{1}{2}\frac{1}{s+4} - \frac{1}{2}\frac{1}{s+6}\end{aligned}$$

Inverse Laplace transform

$$\gamma(t) = \frac{1}{2}e^{-4t} - \frac{1}{2}e^{-6t}$$

We note that as the poles have a negative real part, the impulse response tends to zero.

Time Domain Response to Impulse Disturbance