Control Systems I

Linear Quadratic Regulator

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Linear Quadratic Regulator

Goal: Move from state x to the origin. (i.e., keep x 'small')

$$\dot{x} = Ax + Bu$$

Express the 'cost' of being in state x and applying input u with the function

$$l(x, u) = x^T Q x + u^T R u$$

The total 'cost' of following a particular trajectory is then

$$V(x, u) = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt$$

Assume: $R \succ 0$, $Q \succeq 0$ (i.e., R is positive definite, and Q is positive semi-definite)

LQR: Find the 'best' trajectory

$$\min_u V(x(t), u(t))$$

$$\text{s.t. } \dot{x}(t) = Ax(t) + Bu(t)$$

Motivation for LQR

Consider the system

$$\dot{x} = Ax + Bu \qquad\qquad y = Cx$$

Define $Q = C^T C$ and $R = \rho I$. We are minimizing the cost

$$\begin{aligned} V(x,u) &= \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt \\ &= \int_0^\infty y(t)^2 + \rho u(t)^2 \end{aligned}$$

We're minimizing the relative **energy** in the input and output signals

Large ρ $\rightarrow$ small input energy, output weakly controlled
Small ρ $\rightarrow$ large input energy, output strongly controlled

Note: Any minimal solution **must** be stable / take the state to the origin.
Why? Any non-zero steady-state solution will result in an infinite cost $V(x,u)$

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Real motivation

- Works well in practice
- Works seamlessly for multi-input, multi-output systems
- We can solve it (very common motivation in control!)
- Solution is simple, and easy to implement in embedded controller

LQR - Solution

Linear Quadratic Regulator

Consider a linear multivariable system $\dot{x}(t) = Ax(t) + Bu(t)$. Compute control law $u(t) = -Kx(t)$ such that the following performance criterion is minimized

$$\min_u \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt$$
$$\text{s.t. } \dot{x}(t) = Ax(t) + Bu(t)$$

where R is positive definite, and Q is positive semi-definite.

Optimal Solution

The optimal controller is $u(t) = -Kx(t)$ where

$$K = R^{-1} B^T P$$

and $P = P^T \succ 0$ is the solution of the following **Riccati Equation**

$$A^T P + PA - PBR^{-1} B^T P + Q = 0$$

Proof Sketch

We want to minimize the function

$$J_{LQR} = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt$$

Idea[1]:

We will first write it as

$$J_{LQR} = J_0 + \int_0^\infty (u(t) - u_0(t))^T R (u(t) - u_0(t)) dt$$

for some u_0, where J_0 does not depend on the control law chosen.

From this we can see that the optimal input is $u(t) = u_0(t)$.

[1] We're following the proof of Joao P. Hespanha to avoid variational analysis

Feedback Invariants

A quantity is called a **feedback invariant** if its value does not depend on the choice of the control input $u(t), t \geq 0$.

Lemma: Feedback Invariant

Let P be a symmetric matrix. For every control input $u(t)$, $t \in [0, \infty]$ for which $x(t) \to 0$ as $t \to \infty$, we have that

$$\int_0^\infty x^T(A^TP + PA)x + 2x^TPBudt = -x(0)^TPx(0)$$

Proof:

$$\begin{aligned}
\int_0^\infty x^T(A^TP + PA)x + 2x^TPBudt &= \int_0^\infty (x^TA^T + u^TB^T)Px + x^TP(Ax + Bu)dt \\
&= \int_0^\infty \dot{x}^TPx + x^TP\dot{x}dt \\
&= \int_0^\infty \frac{d(x^TPx)}{dt}dt \\
&= \lim_{t\to\infty} x^T(t)Px(t) - x(0)^TPx(0)
\end{aligned}$$

Square Completion

$$J_{LQR} = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt$$

Add and subtract our feedback invariant

$$= x(0)^T P x(0) + \int_0^\infty x^T Q x + u^T R u + x^T (A^T P + PA) x + 2x^T P B u dt$$

Re-write the terms involving u

$$u^T R u + 2x^T P B u = (u - u_0)^T R (u - u_0) - x^T P B R^{-1} B^T P x$$

where $u_0 = -R^{-1} B^T P x$

Which gives us

$$J_{LQR} = x(0)^T P x(0) + \int_0^\infty x^T Q x + x^T (A^T P + PA) x - x^T P B R^{-1} B^T P x dt$$
$$+ \int_0^\infty (u - u_0)^T R (u - u_0) dt$$

Square Completion

$$J_{LQR} = x(0)^T P x(0) + \int_0^\infty x^T Q x + x^T (A^T P + PA) x - x^T P B R^{-1} B^T P x dt$$
$$+ \int_0^\infty (u - u_0)^T R (u - u_0) dt$$

We see that the optimal solution is

$$u = u_0 = -Kx \qquad\qquad K = R^{-1} B^T P$$

and

$$Q + A^T P + PA - PBR^{-1} B^T P = 0$$

Example

Compute linear controller to minimize the closed-loop performance metric

$$Q = q^4 C^T C \qquad\qquad R = 1$$

for the system $G(s) = \frac{1}{s^2}$, whose state-space representation is

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Example

Solve the ARE $A^T P + PA - PBR^{-1}B^T P + Q = 0$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ - \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & q^4 \end{bmatrix} = 0$$

This results in four equations

$$\begin{aligned} -P_1^2 + 2P_2 &= 0 \\ P_3 - P_1 P_2 &= 0 \\ P_3 - P_1 P_2 &= 0 \\ q^4 - P_2^2 &= 0 \end{aligned}$$

Example

Solving gives

$$P = \begin{bmatrix} \sqrt{2}q & q^2 \\ q^2 & \sqrt{2}q^3 \end{bmatrix}$$

(Note that we've selected the real, positive definite solution)

The controller is

$$K = R^{-1}B^TP = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}q & q^2 \\ q^2 & \sqrt{2}q^3 \end{bmatrix} = \begin{bmatrix} \sqrt{2}q & q^2 \end{bmatrix}$$

Closed-loop System

The closed-loop system is

$$\dot{x} = (A - BK)x = \begin{bmatrix} -\sqrt{2}q & -q^2 \\ 1 & 0 \end{bmatrix} x$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Which has poles at $q\frac{1}{\sqrt{2}}(-1 \pm i)$

We see that

- As $q \to 0$, the input energy is dominant, and the closed-loop poles become the open-loop poles
- As q becomes large, the output energy is dominant, and the system spends more input energy to bring the input to zero faster

Add Reference Input

Add a reference input to the system

$$\dot{x} = \begin{bmatrix} -\sqrt{2}q & -q^2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{N} r$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

We want to steady-state gain between r and y to be one (i.e., $y = r$ at steady-state)

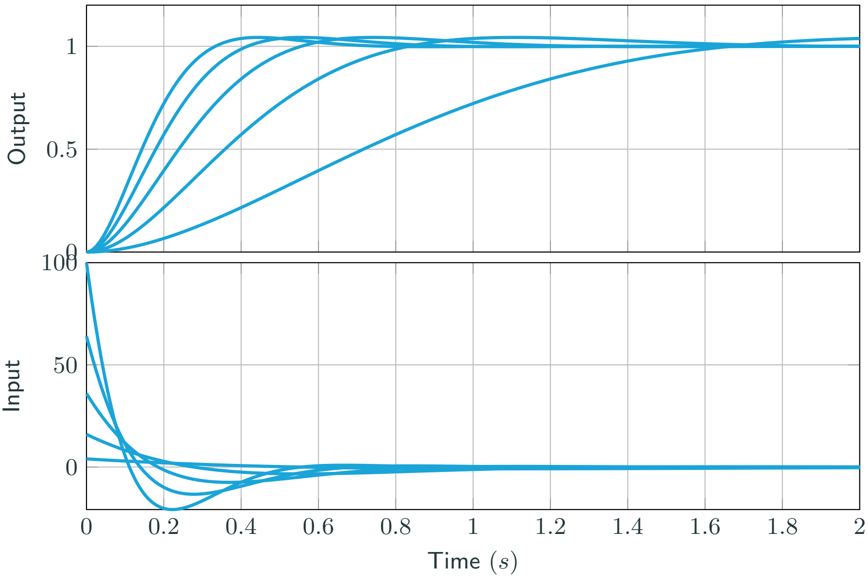
$$0 = \begin{bmatrix} -\sqrt{2}q & -q^2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{N} r \rightarrow x = -\begin{bmatrix} -\sqrt{2}q & -q^2 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{N} r$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -\sqrt{2}q & -q^2 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{N} r$$

We choose $\bar{N}$ such that $y = r$

$$1 = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -\sqrt{2}q & -q^2 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{N}$$
$$= \frac{1}{q^2} \bar{N}$$

So $\bar{N} = q^2$

Step Response



Choice of Weights

Weights are usually determined through a trial-and-error process, but a good initial setting is given by Bryson's rule.

Bryson's rule scales the variables so that the maximum acceptable value for each term is one.

Choose diagonal Q and R with

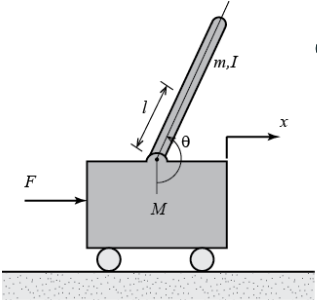
$$Q_{ii} = \frac{1}{\text{maximum acceptable value of } x_i^2}$$
$$R_{jj} = \frac{1}{\text{maximum acceptable value of } u_j^2}$$

Start with Bryson's rule, and then increase or decrease the diagonal values to increase or decrease the convergence rates of the corresponding states.

LQR in Matlab

$$\begin{bmatrix}\dot{x}\\ \ddot{x}\\ \dot{\theta}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & -0.1818 & 2.6727 & 0\\ 0 & 0 & 0 & 1\\ 0 & -0.4545 & 31.1818 & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \theta\\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0\\ 1.8182\\ 0\\ 4.5455\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \theta\\ \dot{\theta}\end{bmatrix}$$



Objectives for a $0.2m$ step in cart position x are:

- Settling time for x and θ of less than 5 seconds
- Rise time for x of less than 0.5 seconds
- Pendulum angle θ never more than 20 degrees (0.35 radians) from the vertical
- Steady-state error of less than 2% for x and θ

Conclusion

- Define the behaviour that we want to achieve via a **value function**
- Choose the control law that minimizes the value function
- LQR is very effective because:
  - The optimal controller is linear
  - The value function is intuitive to tune