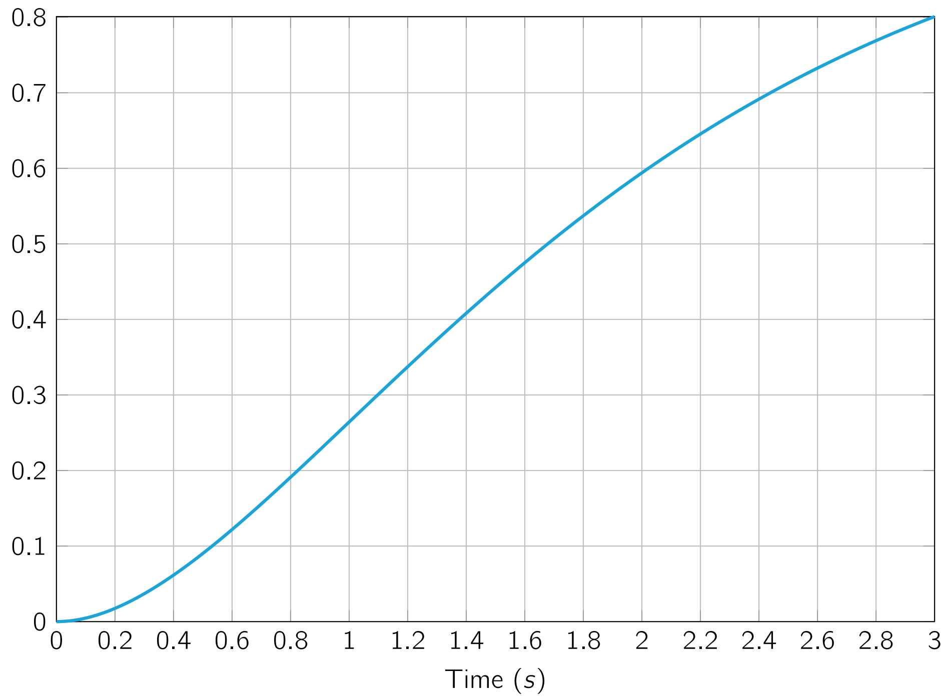
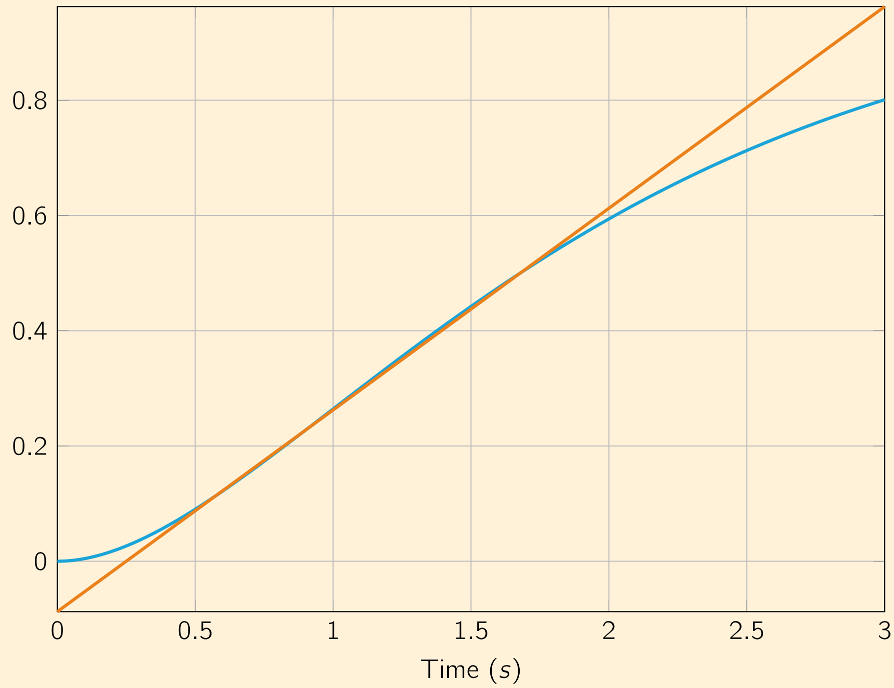
Problem 1.

Your goal is to design a PI controller for the system below using Ziegler-Nichols' method.

$$G(s) = \frac{1}{s^2 + 2s + 1}$$

a) The first three seconds of the step response of the system G is shown below





Measuring the maximum slope and intercept gives

$a = 0.35$ $L = 0.25$

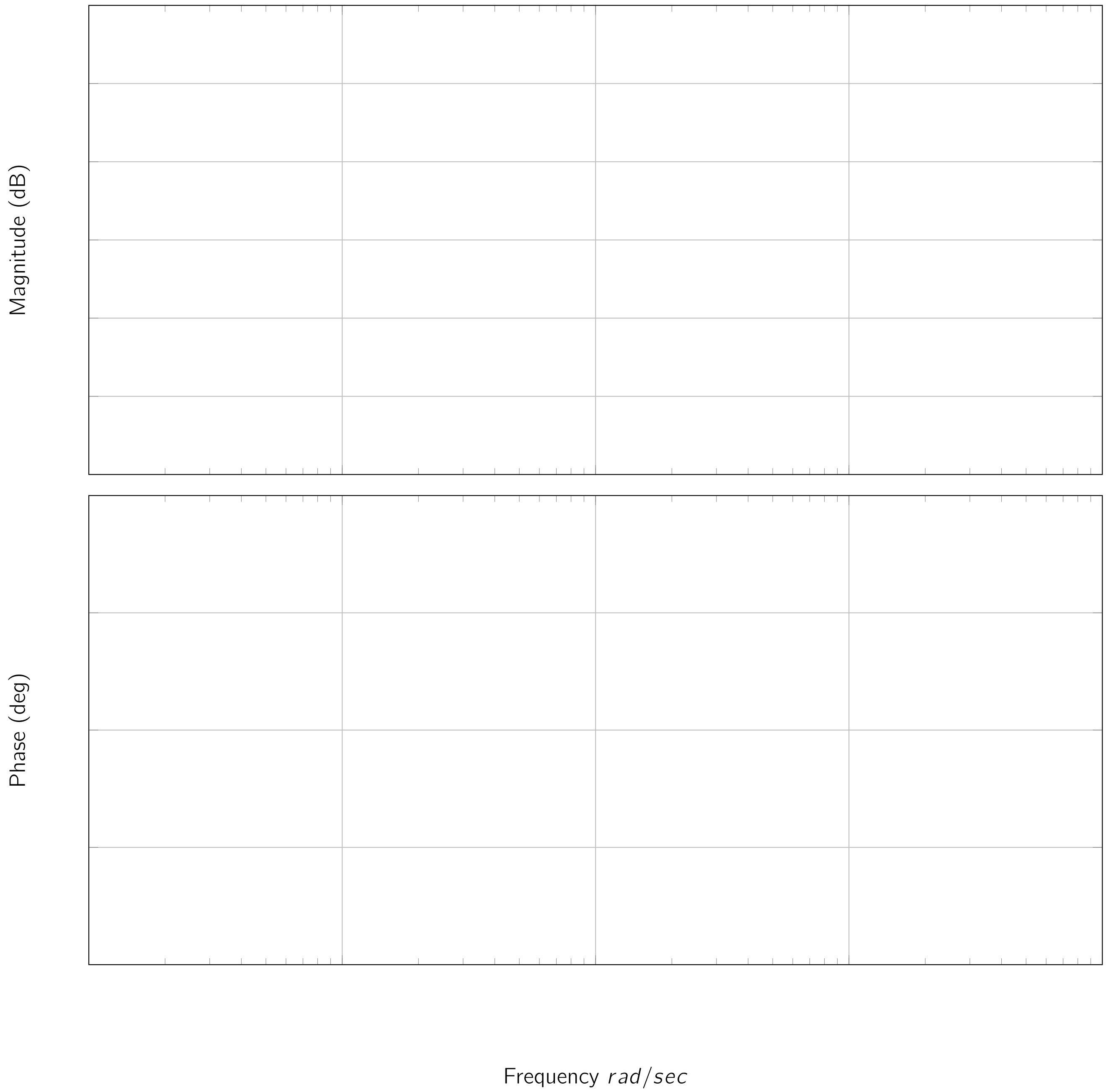
Using the ZN formula from the cheat sheet gives:

$K_p = \frac{0.9}{aL} = \frac{0.9}{0.35 \cdot 0.25} \approx 10$ $T_i = 3.3L = 3.3 \cdot 0.25 \approx 0.8$

Use the Zeigler-Nichols method to compute a PI controller $K(s) = K_p \left(1 + \frac{1}{T_i s}\right)$ for this system

$K_p = \boxed{10}$ $T_i = \boxed{0.8}$

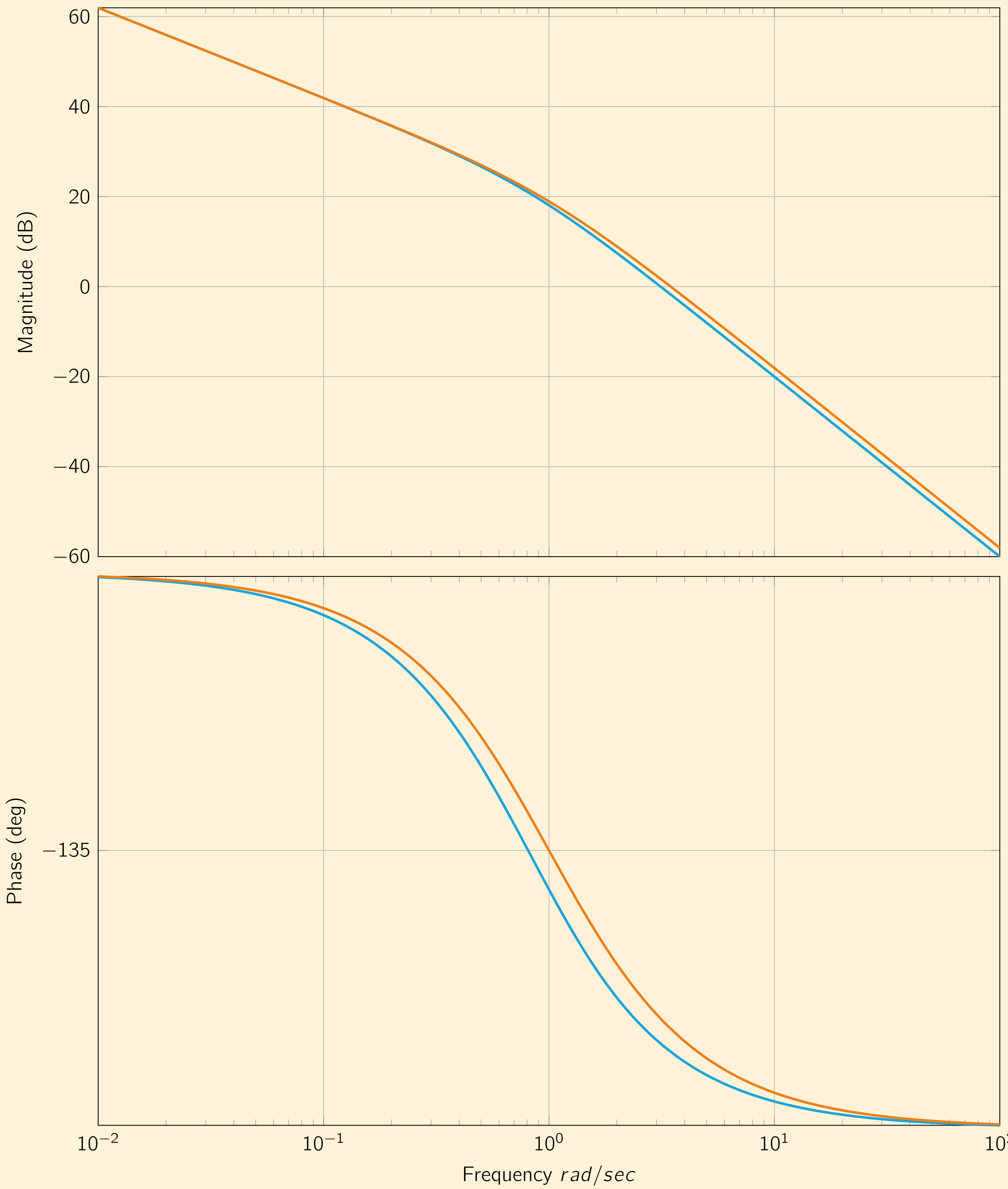
b) Sketch the Bode plot of the open-loop gain $K(s)G(s)$ of the resulting controller and system



The open-loop gain is

$$\begin{aligned} KG &= 10\left(1+\frac{1}{0.8s}\right)\frac{1}{s^2+2s+1} \\ &= 12.5\cdot\frac{1}{s}\cdot\frac{s/1.25+1}{(s+1)^2} \end{aligned}$$

We have almost a pole-zero cancellation, which means we're approximately sketching $12.5\frac{1}{s(s+1)}$, which is shown below (orange = approximate, blue = exact).



Estimate the gain margin and phase margin

$GM =$ $\boxed{\infty}$ $PM =$ $\boxed{14°}$

Phase margin estimate can be very inaccurate and still be correct, as long as it's based on a reasonable plot and analysis of the plot.

c) Give the gain K' such that the system $K' \cdot K(s)G(s)$ will have a phase margin of 45°

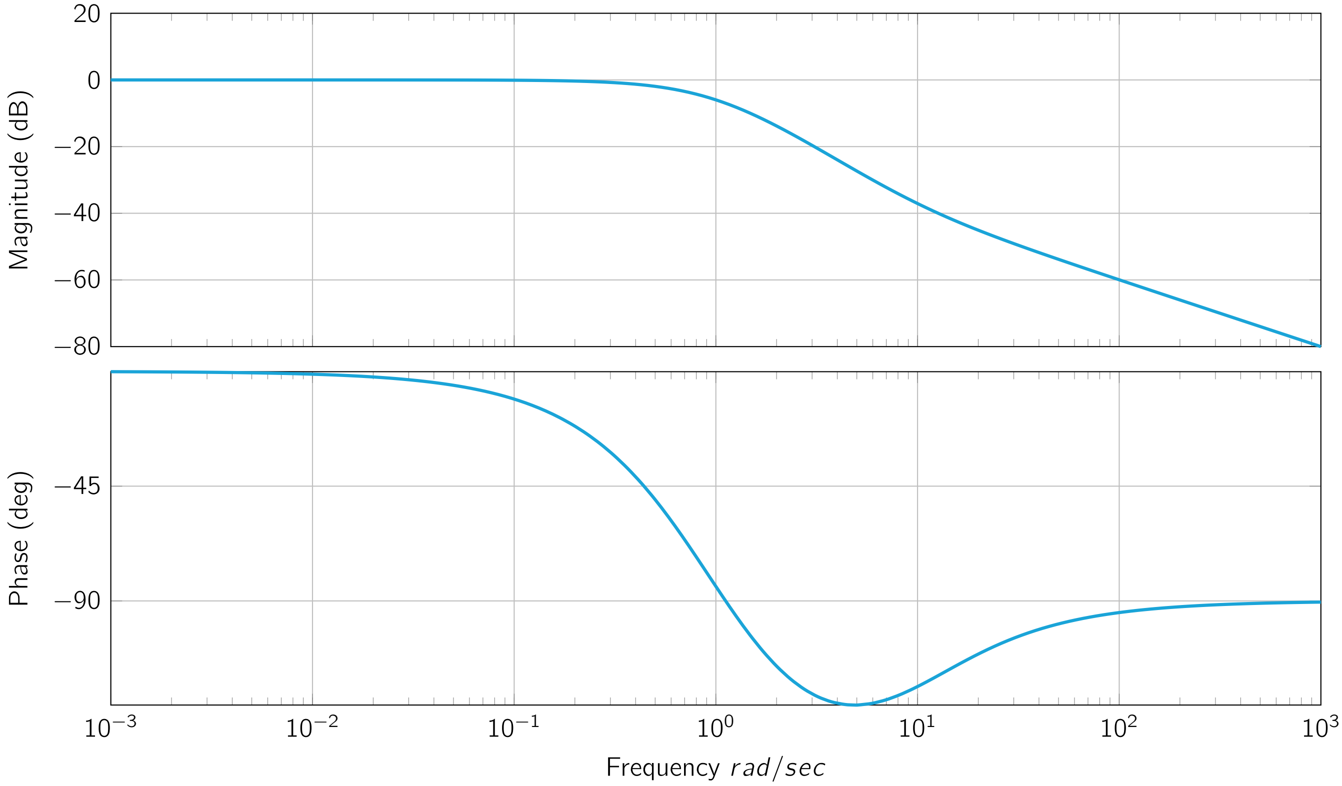
$K' =$ $\boxed{0.1}$

From the Bode plot, we can see that the crossover frequency should be around $1r/s$ for a phase of $-135°$. We can either estimate the gain $K' = -20dB = 0.1$ from the magnitude plot, or we could calculate directly

$$\begin{aligned} K' &= \left| \frac{1}{K(j)G(j)} \right| \\ &= \left| \frac{1}{12.5} \cdot j \cdot \frac{(j+1)^2}{j/1.25+1} \right| \\ &= 0.08 \cdot 2 \cdot 0.78 \\ &= 0.125 \end{aligned}$$

Problem 2.

a) Estimate the transfer function for the bode plot given below



$$G(s) = \boxed{\frac{s/10+1}{s^2+2s+1} = \frac{s/10+1}{(s+1)^2}}$$

b) Give the model in control canonical form

$$\dot{x} = \boxed{\begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}} x + \boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} u$$

$$y = \boxed{\begin{bmatrix} 1/10 & 1 \end{bmatrix}} x + \boxed{0} u$$

c) Design a state-feedback controller so that the closed-loop system will have an overshoot of 25% and a settling time of 1 second.

The overshoot and settling time conditions give $\zeta = 0.4$ and $\omega_n = 9.9$, giving a target characteristic equation of

$$\alpha(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 7.9s + 98$$

We're in control canonical form, so the control law is

$$\begin{aligned} K &= \begin{bmatrix} -a_1 + \alpha_1 & -a_2 + \alpha_2 \end{bmatrix} \\ &= \begin{bmatrix} -2 + 7.9 & -1 + 98 \end{bmatrix} \\ &= \begin{bmatrix} 5.9 & 97 \end{bmatrix} \end{aligned}$$

$$K = \boxed{\begin{bmatrix} 5.9 & 97 \end{bmatrix}}$$

d) Design an observer with poles at -10 and -20

$$\begin{aligned} \det(sI - (A - LC)) &= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 1/10 & 1 \end{bmatrix} \\ &= \left| \begin{bmatrix} s + 2 + L_1/10 & 1 + L_1 \\ -1 + L_2/10 & s + L_2 \end{bmatrix} \right| \\ &= (s + 2 + L_1/10)(s + L_2) - (1 + L_1)(-1 + L_2/10) \\ &= s^2 + (L_1/10 + L_2 + 2)s + L_1 + 1.9L_2 + 1 \\ &= \alpha(s) = (s + 10)(s + 20) = s^2 + 30s + 200 \end{aligned}$$

Solving for L gives

$$L = \begin{bmatrix} 180 \\ 10 \end{bmatrix}$$

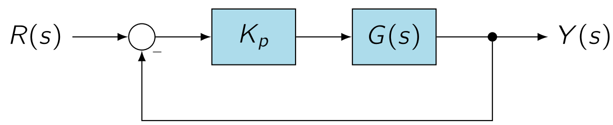
$$L = \boxed{\begin{bmatrix} 180 \\ 10 \end{bmatrix}}$$

e) Under what circumstances would it be better to take the poles of the observer to be -100 and -200?

The observer needs to be faster than the system, so these poles would be fine.
If the sensor produces noise between the frequencies of -10 and -100, then we should not take the faster poles, otherwise it's better.

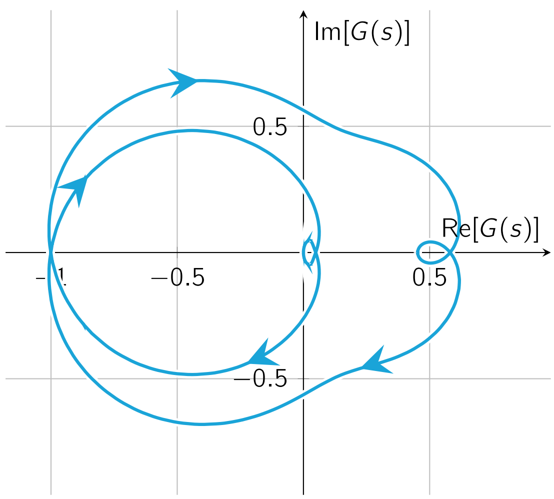
Problem 3.

a) Determine the range of gains K_p for which the closed-loop system below will be stable.



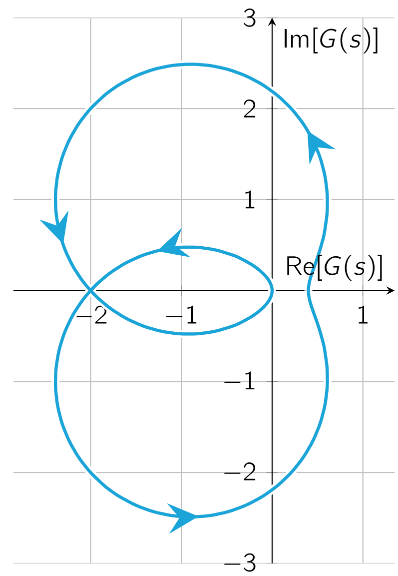
i) $K_p \in$ $[-1.7, 1]$

The system whose Nyquist plot is shown below is open-loop stable.

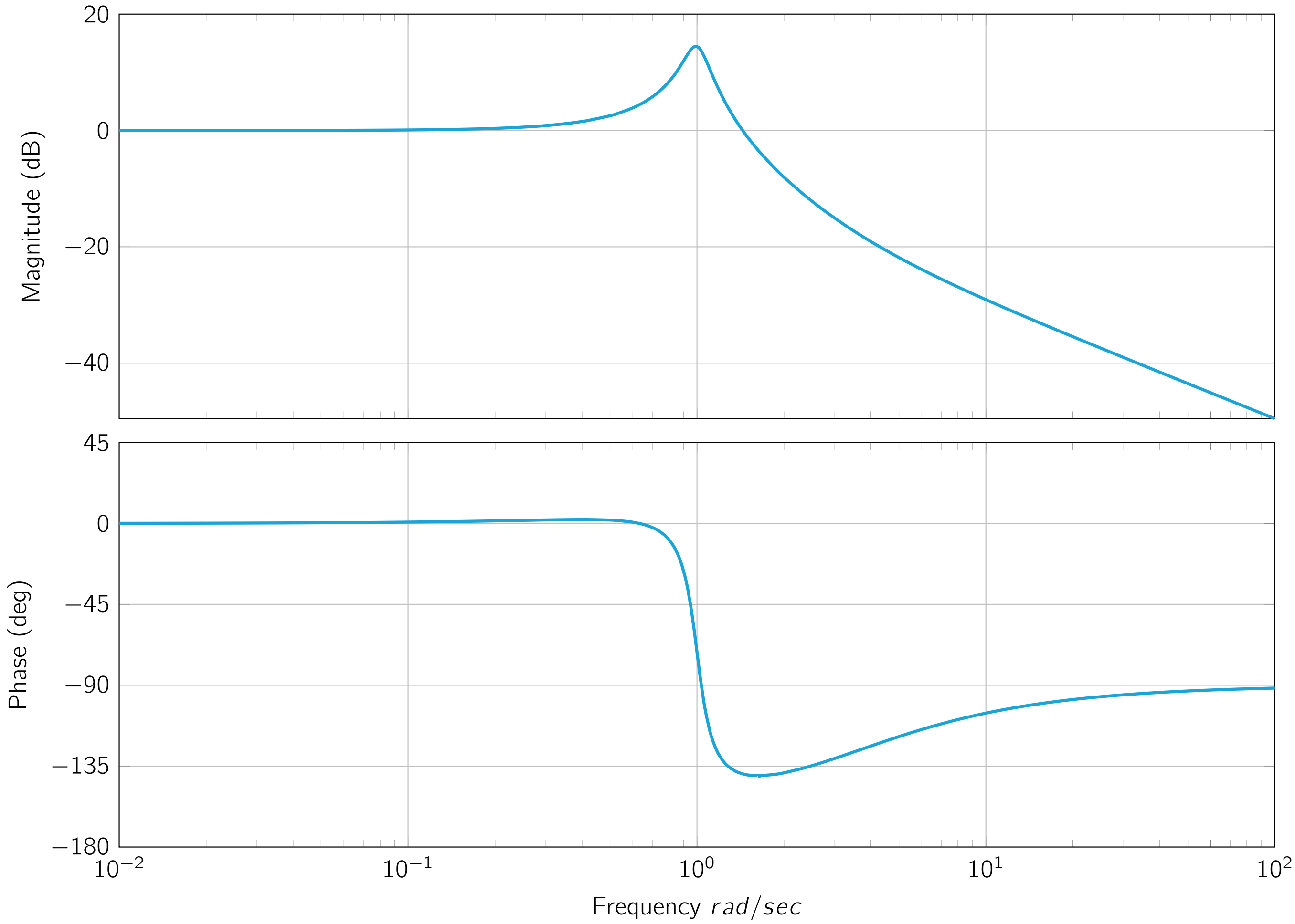


ii) $K_p \in$ $[0.5, \infty]$

$$G(s) = 0.4\frac{s+1}{s^2 - 0.2s + 1}$$



b) Consider the Bode plot shown below



i) What are the stability margins of this system?

Gain margin = ∞ Phase margin = 40

ii) What will the stability margins be if the controller $K(s) = 1 + \frac{1}{T_i s}$ is used with $T_i = 10$?

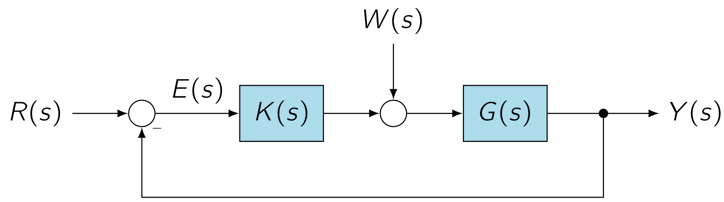
Gain margin = ∞ Phase margin = 40

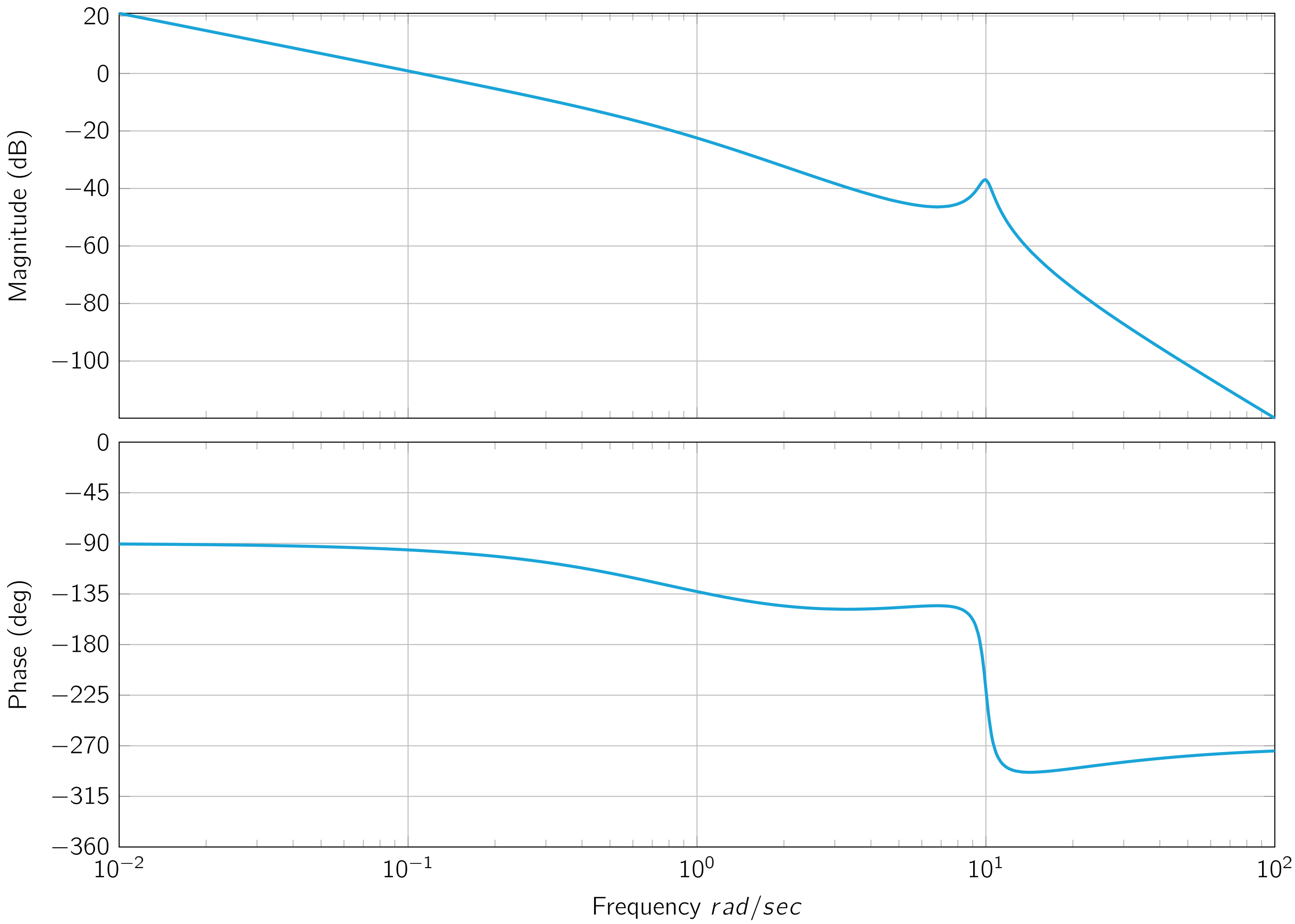
iii) Give two methods to increase the phase margin

Change the gain to any other value or add a lead compensator

Problem 4.

Consider the following closed-loop system and the bode plot of $G(s)$ given below.





a) What is the steady-state error of the closed-loop system for $K(s) = 10$ for the following situations:

i) $W(s) = 0$ and $R(s) = \frac{1}{s}$ $\qquad \lim_{t\to\infty} e(t) = \boxed{0}$

ii) $W(s) = \frac{1}{s}$ and $R(s) = 0$ $\qquad \lim_{t\to\infty} e(t) = \boxed{-0.1}$

iii) $W(s) = \frac{2}{s}$ and $R(s) = \frac{1}{s^2}$ $\qquad \lim_{t\to\infty} e(t) = \boxed{0.8}$

The transfer function to the error is

$$\begin{aligned} E &= R - G(W + K * E) \\ E(1+GK) &= R - GW \\ E &= \frac{1}{1+GK}R - \frac{G}{1+GK}W \end{aligned}$$

The low-frequency system looks like $\frac{1}{10s}$, and the controller is $K = 10$

$$\begin{aligned} E &= \frac{1}{1+\frac{1}{10s}10}R - \frac{\frac{1}{10s}}{1+\frac{1}{10s}10}W \\ &= \frac{1}{1+\frac{1}{s}}R - \frac{\frac{1}{10s}}{1+\frac{1}{s}}W \\ &= \frac{s}{s+1}R - \frac{\frac{1}{10}}{s+1}W \end{aligned}$$

Use the final value theorem to compute the result in each case

i) $W(s) = 0$, $R(s) = 1/s$

$$\begin{aligned} \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) &= s\frac{s}{s+1}R - s\frac{\frac{1}{10}}{s+1}W \\ &= s\frac{s}{s+1}\frac{1}{s} \\ &= 0 \end{aligned}$$

ii) $W(s) = 1/s$, $R(s) = 0$

$$\begin{aligned} \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) &= s\frac{s}{s+1}R - s\frac{\frac{1}{10}}{s+1}W \\ &= -s\frac{\frac{1}{10}}{s+1}\frac{1}{s} \\ &= -0.1 \end{aligned}$$

iii) $W(s) = 2/s$, $R(s) = 1/s^2$

$$\begin{aligned} \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) &= s\frac{s}{s+1}R - s\frac{\frac{1}{10}}{s+1}W \\ &= s\frac{s}{s+1}\frac{1}{s^2} - s\frac{\frac{1}{10}}{s+1}\frac{2}{s} \\ &= 1 - 0.2 \\ &= 0.8 \end{aligned}$$

b) Design a lag-compensator $K(s)$ to reduce all steady-state errors listed above to zero, maximize the bandwidth and keep the overshoot in response to a step reference below 25%.

$$K(s) = \boxed{10\left(1 + \frac{1}{5s}\right)}$$

We need to increase the system type by adding an integrator. We'll design a phase-lag compensator with $\alpha = \infty$; i.e., a PI controller

$$K(s) = K_p\left(1 + \frac{1}{T_i s}\right)$$

Step response less than 25% $\rightarrow$ Phase margin better than 45°, or a phase of $-135°$.
Since a lag compensator is only gong to reduce the phase, we look for the highest frequency where the phase is higher than $-135°$, which is at $1r/s$. The gain is approx $-20dB$ at this frequency, so we choose the gain to be $K_p = 20dB = 10$.
Since we have chosen the phase margin to be exactly 45°, we need to be conservative with the lag compensator, and choose the corner frequency to be 1/5 the crossover frequency : $1/T_i = 1/5 \rightarrow T_i = 5$.
Alternatively, we could choose the phase margin to be about 60°, resulting in a crossover frequency of $0.6r/s$ and a gain of $K_p = 16dB = 6.3$. The resulting corner frequency would then be around $1/T_i = 0.6/3 \rightarrow T_i = 5$.

c) Give a difference equation using the Tustin approximation for your controller with a sample period of $T_s = 0.1s$.

$u(k) = $ $u(k-1) + 10.1y(k) - 9.9y(k-1)$

We replace $s \rightarrow \frac{2}{T_s}\frac{z-1}{z+1} = 20\frac{z-1}{z+1}$

$$\begin{aligned}
K(z) &\approx K_p\left(1 + \frac{1}{T_i 20\frac{z-1}{z+1}}\right)\\
&= \frac{K_p}{20T_i}\frac{(1+20T_i)z + 1 - 20T_i}{z-1}\\
&= \frac{K_p}{20T_i}\frac{(1+20T_i) + (1-20T_i)z^{-1}}{1-z^{-1}}
\end{aligned}$$

We convert to a difference equation:

$$\begin{aligned}
U(z) &= z^{-1}U(z) + \frac{K_p}{20T_i}(1+20T_i)Y(z) + \frac{K_p}{20T_i}(1-20T_i)z^{-1}Y(z)\\
u(k) &= u(k-1) + \frac{K_p}{20T_i}(1+20T_i)y(k) + \frac{K_p}{20T_i}(1-20T_i)y(k-1)\\
u(k) &= u(k-1) + 10.1y(k) - 9.9y(k-1)
\end{aligned}$$

d) If the controller measures $y(0) = 1$ and $y(0.1) = 2$, and the input and output are zero before time $t = 0$, what will the control action be at $t = 0.1s$?

$u(0.1) = $ 20.4

We just run the difference equation:

$$\begin{aligned}
u(0) &= u(-1) + 10.1y(0) - 9.9y(-1) = 10.1\\
u(1) &= u(0) + 10.1y(1) - 9.9y(0) = 10.1 + 10.1\cdot 2 - 9.9\cdot 1 = 20.4
\end{aligned}$$

Problem 5.

Consider the system $G(s) = \dfrac{6}{s+4}$

a) Give the system in control canonical form

$$\dot{x} = \boxed{-4}\, x + \boxed{1}\, u$$

$$y = \boxed{6}\, x + \boxed{0}\, u$$

b) Augment the system to add an integrator for offset-free tracking.

$$\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \boxed{\begin{bmatrix} -4 & 0 \\ -6 & 0 \end{bmatrix}} \begin{bmatrix} x \\ x_I \end{bmatrix} + \boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}\, u$$

$$y = \boxed{\begin{bmatrix} 6 & 0 \end{bmatrix}} \begin{bmatrix} x \\ x_I \end{bmatrix} + \boxed{\begin{bmatrix} 0 & 0 \end{bmatrix}}\, u$$

c) The augmented system is not observable. Why is this not a problem?

The integrator state is simulated, and so can be measured directly.

d) Design an LQR controller for the augmented system with the weights $Q = \begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix}$ and $R = 1$

We solve the Riccati equation to compute the Lyapunov matrix $P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}$

$$Q + A^T P + PA - PBR^{-1}B^T P = 0$$

$$\begin{bmatrix} -P_1^2 - 8P_1 - 12P_2 + 8 & -4P_2 - 6P_3 - P_1 P_2 \\ -4P_2 - 6P_3 - P_1 P_2 & 1 - P_2^2 \end{bmatrix} = 0$$

Solving gives $P = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

Compute the controller K

$$K = -R^{-1}B^T P = \begin{bmatrix} -P_1 & -P_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \end{bmatrix}$$

$$K = \boxed{\begin{bmatrix} 2 & -1 \end{bmatrix}}$$

e) Design an estimator gain so that the estimator frequency is 4 times faster than the open-loop system.

We need only design an estimator for the system, and not the augmented system
The target frequency is $4 \cdot 4 = 16$

$$A - LC = -4 - L6 = -16 \rightarrow L = (16-4)/6 = 2$$

$L = \boxed{2}$

f) Supposed that the LQR gains are changed to $Q = \begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix}$ and $R = 2$

i) How will the estimator poles change? Why?

They won't change. The controller does not influence the estimator poles.

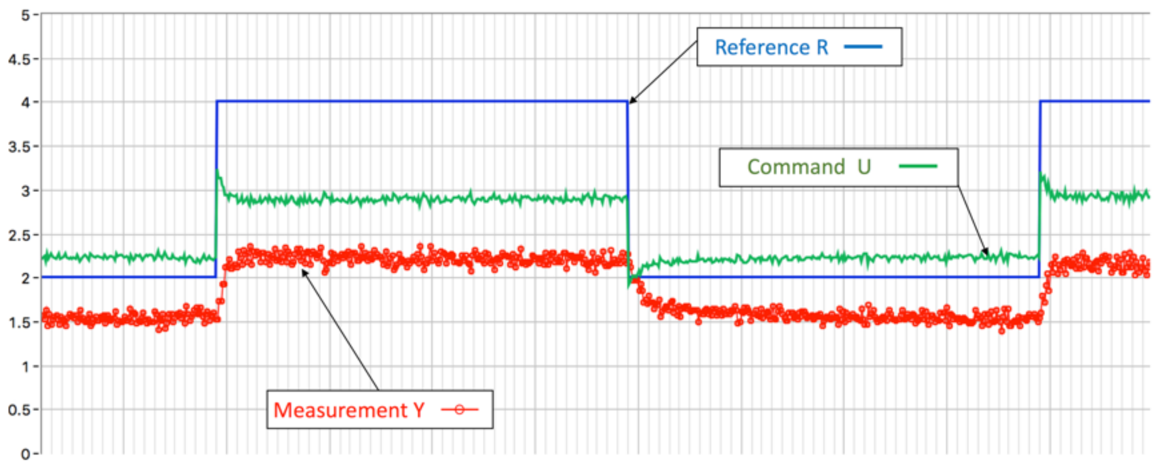
ii) How do you expect the closed-loop poles to change? Why?

The input is now weighted more strongly, meaning that less input authority will be used and the closed-loop system will act more slowly. As a result, the poles will become 'slower', or move 'to the right' towards the imaginary axis.

Problem 6. Travaux Pratiques

Notes:

- The answer and the justification must be correct to get the points.
- Measurements have been made on the electrical drives used during the labs.



1a) A proportional controller + a feedforward command is used. Give the controller parameters.

Proportional gain $K_p =$ 0.5 Feedforward command $U_0 =$ 2 V

Justify your response

$U = K_p(\text{Ref} - \text{Mes}) + U_0$

$2.9 = K_p(4–2.1) + U_0$ $2.2 = K_p(2 - 1.6) + U_0$ => $K_p = 0.5,\ U_0 = 2V$

1b) Given that the friction is 0.4 [V], compute a new feedforward command to follow a reference of 3 [V] without error. Justify your response.

$Y + \text{friction} = U \cdot K$ → $2.2 + 0.4 = 2.9K$ → $K = \sim 0.89$

$3 + 0.4 = U_0 \cdot 0.89$ → $U_0 = \sim 3.8$ [V]

2a) The speed sensor has been replaced, what are the new steady-state (static) gain and time constant of the system?

dMes $= 3.5 - 0 = 3.5V$ $\quad dU = 3 - 0.5 = 2.5V \quad \rightarrow \quad K = dM/dU = 3.5/2.5 = 1.4$

$63\% \rightarrow 3.5 \cdot 0.63 = 2.2V \quad \rightarrow \quad \tau = \sim 0.5s$

2b) Compute a controller so that the closed-loop system matches the model $T(s) = \dfrac{1}{1+\tau_m s}$ with $\tau_m = 0.15$ [s].

$$\frac{1}{1+\tau_m s} = \frac{KG}{1+KG}$$

$$K_p = \frac{\tau}{K \cdot \tau_m} = \frac{0.5}{1.4 \cdot 0.15} = \sim 2.38$$

$$T_i = \tau = 0.5$$