Exercise 1

A bar of unstretched length l_0 with uniform mass density ρ is hung from a rigid ceiling. The bar has a cross-sectional area $A(y)$ which varies along the length of the bar and has a value A_0 at the bottom (see figure 1).

a) What is the equation that describes $A(y)$ if the bar is to have uniform stress σ in the horizontal plane along its length?

b) What is the total elongation of the bar in this case?

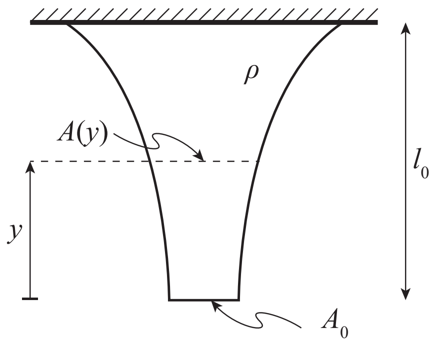

Figure 1: Structure with nonuniform cross–section supported on ceiling.

Exercise solution 1

Given: Unstretched length l_0, mass density ρ, cross section area of the bottom A_0, and uniform stress σ.

Asked: Find the equation to describe $A(y)$ and the total elongation of the bar.

Relevant relationships:

Sum of forces

$$\sum_i F_i = 0 \qquad i \in x, y, z$$

Definition of stress

$$\sigma = \frac{P}{A}$$

a)

Variation 1 Start with a horizontal section at point y from the bottom. Use h as the variable for the coordinate in the section. We can find $W(y)$ by integrating over h from 0 to y:

$$W(y) = \int_0^y g\rho A(h)\,\mathrm{d}h = \rho g \int_0^y A(h)\,\mathrm{d}h$$

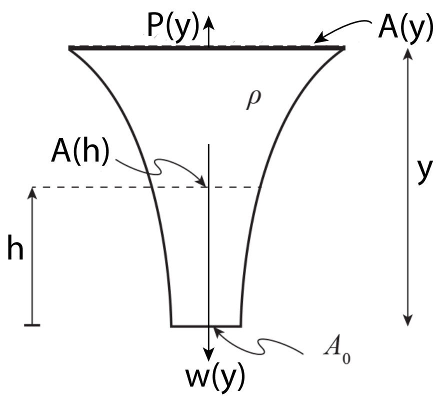

Figure 2: Forces on the structure.

from the equilibrium of forces:

$$\sum Fy = 0 \quad \rightarrow \quad P(y) - W(y) = 0, \quad P(y) = W(y)$$

where

$$P(y) = \rho g \int_0^y A(h)\,\mathrm{d}h$$

Since the stress over the structure is supposed to be constant ($\sigma(y) = \sigma$) we find

$$\sigma = \frac{P(y)}{A(y)} \quad \rightarrow \quad \sigma = \frac{\rho g \int_0^y A(h)\,\mathrm{d}h}{A(y)}$$

$$\sigma A(y) = \rho g \int_0^y A(h)\,\mathrm{d}h \quad \rightarrow \quad A(y) = \frac{\rho g}{\sigma} \int_0^y A(h)\,\mathrm{d}h$$

This is the integral form of a common equation. To convert it into a more familiar form, we can differentiate both sides with respect to y

$$\frac{\mathrm{d}A(y)}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\rho g}{\sigma}\int_0^y A(h)\,\mathrm{d}h\right) = \frac{\rho g}{\sigma}\frac{\mathrm{d}}{\mathrm{d}y}\left(\int_0^y A(h)\,\mathrm{d}h\right)$$

From the fundamental theorem of calculus we get

$$\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y A(h)\,\mathrm{d}h = A(y) \quad \rightarrow \quad \frac{\mathrm{d}A(y)}{\mathrm{d}y} = \frac{\rho g}{\sigma} A(y)$$

which is a first order ordinary differential equation. We use the Ansatz $A(y) = ce^{ky}$ where c and k are constants to be determined.
This gives us

$$ck \cdot e^{ky} = \frac{\rho g}{\sigma} c \cdot e^{ky} \quad \rightarrow \quad k = \frac{\rho g}{\sigma}$$

Using the boundary condition $A(0) = A_0$ we get

$$A(0) = ce^{k \cdot 0} = c \quad \rightarrow \quad c = A_0$$

which finally gives us

$$A(y) = A_0 e^{\left(\frac{\rho g}{\sigma} y\right)}$$

Variation 2 Alternatively, we can resolve this problem by solving the microscopic equilibrium equation as seen in the course. In this case, we consider a slice of height dy:

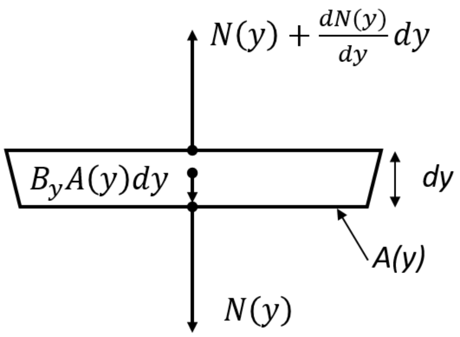

Figure 3: Forces on the structure.

Since the stress over the structure is supposed to be constant ($\sigma(y) = \sigma$), the kinematic equation becomes:

$$\frac{\mathrm{d}u(y)}{\mathrm{d}y} = \varepsilon(y) = \frac{\sigma}{E} = const$$

The force acting at the bottom of the slice is:

$$N(y) = A(y)\sigma$$

From the force equilibrium in the y-direction, we obtain:

$$\frac{\mathrm{d}N(y)}{\mathrm{d}y} + N(y) - \rho g A(y) - N(y) = 0$$

Upon substitution of $N(y)$, we get:

$$\frac{\mathrm{d}}{\mathrm{d}y} A(y)\sigma = \rho g A(y)$$

Which simplifies to the formula we solved in variation 1:

$$\frac{\mathrm{d}A(y)}{\mathrm{d}y} = \frac{\rho g}{\sigma} A(y)$$

b)

We can calculate the elongation using Hooke's Law ($\sigma = E\varepsilon$) and $\varepsilon = \delta/l_0$, therefore

$$\sigma = E\frac{\delta}{l_0}, \quad \delta = \frac{\sigma l_0}{E}$$

since σ is a constant, as is l_0 and E, we are done.

Exercise 2

You want to calculate the deformation of a Y shaped trabecula section in the trabecular bone of a vertebra as shown in figure 4. To simplify the calculation, we model the Y-shaped trabecula as shown in figure 5. Assume that the horizontal beam is both infinitely thin and stiff and that it does not bend.

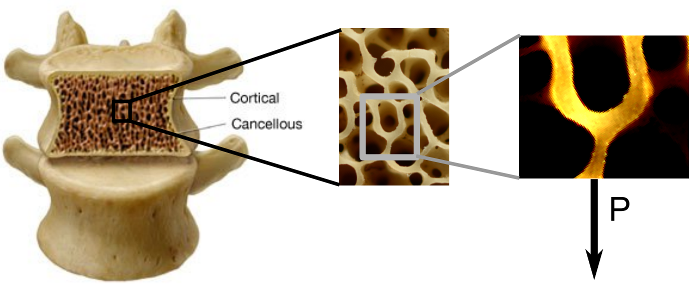

Figure 4: Schematics of a trabecular bone.

1. A force of $F1 = 0.5\,\mathrm{N}$ is applied to the trabecular bone substructure. Calculate the total elongation of the substructure, given the lengths $L_1 = 1.5\,\mathrm{mm}$, $L_2 = 0.8\,\mathrm{mm}$ and the diameter $d = 200\,\mu\mathrm{m}$. Young's modulus of the trabecular bone can be assumed to be $E = 22\,\mathrm{GPa}$.

2. There is now an extra force $F2 = 0.2\,\mathrm{N}$ applied to the bone as shown in figure 5. State the superposition principle and calculate the total elongation.

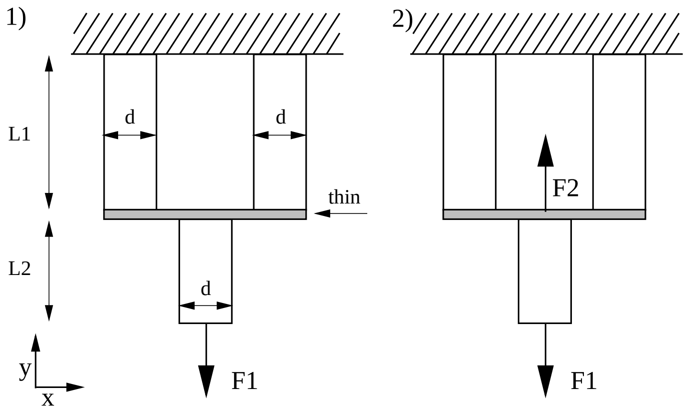

Figure 5: Simplification of the bone for calculations.

Exercise solution 2

Given: Forces $F1$ and $F2$, Young's modulus E, diameter and lengths of the substructures

Asked: Total elongation δ_{tot} in two different cases

Relevant relationships:

Hook's law for normal stress $\sigma = E \cdot \varepsilon$

Normal stress $\varepsilon = \dfrac{\Delta L}{L} = \dfrac{\delta}{L}$

Normal strain $\sigma = \dfrac{P}{A}$

1. In the first question, there is only one force, F_1. The total elongation of the structure is the sum of the elongations from each substructure.
   The elongation δ can be expressed as a function of the applied force P by using Hooke's law and the definitions of the normal stress σ and the normal strain ε : $\delta = L\varepsilon = L\dfrac{\sigma}{E} = \dfrac{LP}{EA}$
   For the lower part, the length of the segment is L_2, and the applied force F_1 is distributed on the area $A = \pi({}^{d}\!/_{2})^2$. The same force F_1 is acting on the upper part of the structure (think about the methods of section if you are not convinced), only that the force is distributed on the total area of the two segments (with the length L_1). Therefore, the area is $2A\ = 2\pi({}^{d}\!/_{2})^2$.

   As indicated, we neglect the interconnection structure and the mass of each segment. The Young's modulus E is the same for all segments.

$$\begin{aligned}\delta_{\text{tot},1} &= \frac{L_1F_1}{E\cdot 2A} + \frac{L_2F_1}{EA}\\ &= \frac{1.5\cdot 10^{-3}\cdot 0.5}{22\cdot 10^{9}\cdot 2\cdot \pi(\frac{0.2\cdot 10^{-3}}{2})^2} + \frac{0.8\cdot 10^{-3}\cdot 0.5}{22\cdot 10^{9}\cdot \pi(\frac{0.2\cdot 10^{-3}}{2})^2}\\ &= 1.12\times 10^{-6}\,\text{m}\end{aligned}$$

The total elongation of the structure is therefore approximately 1.12×10^{-6} m.

2. There are now two forces applied to the system, F_1 and F_2. The superposition principle states that, for all **linear** systems, the response to several stimuli (forces in our case) is the sum of all the responses that would have been caused by each individual stimulus. As a consequence, the elongation of the system is the sum of the elongation $\delta_{tot,1}$ caused by F_1 alone and the elongation $\delta_{tot,2}$ caused by F_2 alone. We already determined $\delta_{tot,1}$ in the previous question. Only the top part of the bone will be deformed due to F_2 (think about the methods of section if you are not convinced). Since the F_2 is pointing upwards, its effect is actually a contraction of the beams. The total deformation due to F_2 alone is thus :

$$\begin{aligned}\delta_{\text{tot},2} &= -\frac{L_1F_2}{E\cdot 2A} + 0\\ &= -\frac{1.5\cdot 10^{-3}\cdot 0.2}{22\cdot 10^{9}\cdot 2\cdot \pi(\frac{0.2\cdot 10^{-3}}{2})^2}\\ &= -4.34\times 10^{-7}\,\text{m}\end{aligned}$$

Finally, according to the superposition principle, the total elongation due to both F_1 and F_2 is $\delta_{tot} = \delta_{tot,1} + \delta_{tot,2} = (1.12 - 0.43)10^{-7}m = 0.69\,\mu\text{m}$.

Exercise 3

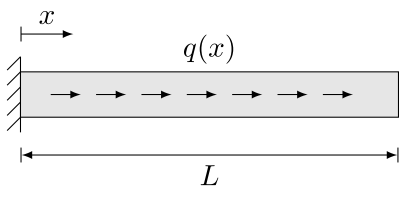

Figure 6: Beam with a distributed load.

The bar in figure 6 is loaded with a force that is distributed over the length of the beam. The load is described as

$$q(x) = q_0\cdot\frac{x}{L} + q_1$$

and we want to calculate the internal forces $N(x)$ and the displacement field $u(x)$ along the beam.

1. Find the differential equation that describes the displacement field of the beam $u(x)$ as a function of E, A and $q(x)$, by considering the three essential equations of structural mechanics:
   - constitutive equation : $E = \frac{\sigma}{\varepsilon}$
   - kinematic equation : $\varepsilon(x) = \dfrac{\partial u(x)}{\partial x}$
   - equilibrium equation : $\dfrac{\partial N(x)}{\partial x} + \sum_i q_i(x) + B_x A(x) = 0$
2. Find the boundary conditions for the bar and deduce boundary conditions for u (or its derivatives).
3. Solve the equations for $u(x)$. Deduce the expression of the internal force in the beam $N(x)$.

Exercise solution 3

Given: Geometry, A, E. Distributed load $q(x) = q_0 \cdot \frac{x}{L} + q_1$

Asked: Governing differential equation, boundary conditions, internal force $N(x)$ and displacement field $u(x)$.

Relevant relationships:

Equilibrium equation

$$\frac{\partial N(x)}{\partial x} + \sum_i q_i(x) + B_x A(x) = 0$$

Constitutive equation (Hooke's law)

$$E = \frac{\sigma}{\varepsilon}$$

Kinematic equation

$$\varepsilon(x) = \frac{\partial u(x)}{\partial x}$$

1. Differential equation that links u and q Since $B_x = 0$, the equilibrium equation becomes, in our case, $\dfrac{\partial N(x)}{\partial x} + q(x) = 0$.

Comment : this can also be demonstrated as during the class by considering an infinitesimla element. From the equilibrium of forces in x-direction on a differential element (figure 7) and using the linear approximation $N(x + dx) = N(x) + \dfrac{\partial N(x)}{\partial x} \cdot dx$, we get the same result :

$$-N(x) + N(x) + \mathrm{d}N(x) + q(x) \cdot \mathrm{d}x = 0 \quad \rightarrow \quad \frac{\partial N(x)}{\partial x} + q(x) = 0$$

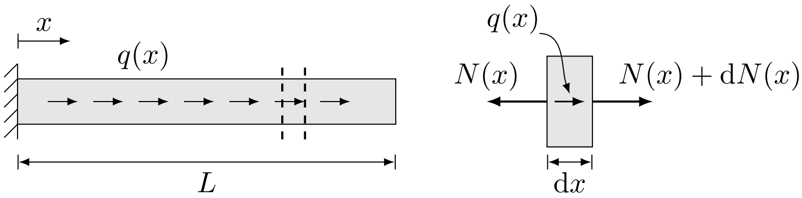

Figure 7: Beam with distributed load and differential element.

From Hooke's law we get:

$$E(x) = \frac{\sigma(x)}{\varepsilon(x)} = \frac{N(x)}{A(x)\varepsilon(x)} \to N(x) = EA\,\varepsilon(x) \tag{1}$$

where E and A are constant.

By substituting this into the equilibrium equation we get:

$$\frac{\partial N(x)}{\partial x} = \frac{\partial AE \cdot \varepsilon(x)}{\partial x} = -q(x)$$

Finally we can add in the kinematic equation:

$$\frac{\partial}{\partial x}\left(AE \cdot \frac{\partial u(x)}{\partial x}\right) = -q(x)$$

Since A and E are constant, we can rewrite this as:

$$AEu''(x) = -q(x) \tag{2}$$

2. Boundary conditions The bar is clamped at $x = 0$ so there is no displacement at that point: $u(0) = 0$. The free end can not support any force, so the second boundary condition is $N(L)0$, which becomes $AEu'(L) = 0$ by using the kinematic equation.

3. Displacement field u(x) We are now looking to solve our differential equation for the displacement field.

Eqn. (2) becomes, if we replace $q(x)$ by its explicit expression :

$$AE \cdot \frac{\partial^2 u(x)}{\partial x^2} = -q_0 \frac{x}{L} - q_1$$

We have to integrate twice to find our solution:

$$AE \cdot \frac{\partial u}{\partial x}(x) = -q_0 \frac{x^2}{2L} - q_1 x + C_1$$

$$u(x) = \frac{1}{AE}\left(-q_0 \frac{x^3}{6L} - \frac{q_1}{2}x^2 + C_1 x + C_2\right)$$

where C_1 and C_2 are coefficients to be determined with the boundary conditions.

For C_2 we get:

$$u(0) = 0 \rightarrow C_2 = 0$$

For C_1 we have:

$$N(L) = 0 \rightarrow u'(L) = 0.$$

$$AE \cdot \frac{\partial u(L)}{\partial x} = -q_0 \frac{L^2}{2L} - q_1 L + C_1 = 0 \quad \rightarrow \quad C_1 = \frac{q_0 L}{2} + q_1 L$$

Thus we finally obtain our solution:

$$u(x) = \frac{1}{AE}\Big(-q_0 \frac{x^3}{6L} - \frac{q_1}{2}x^2 + (\frac{q_0 L}{2} + q_1 L)\, x\Big)$$

4. Internal force N(x) Let's now determine the internal force N(x).

From eqn. (1) we get:

$$N(x) = AE\varepsilon(x) = AEu'(x)$$

By integrating eqn. (2) once, we can determine $u'(x)$. Here we integrate from L to x because we know the boundary condition at $x = L$:

$$u'(x) - u'(L) = \int_L^x \frac{-q(x)}{AE} dx$$

$$u'(x) = \int_L^x \frac{-q(x)}{AE} dx + u'(L)$$

$$N(x) = AEu'(x) = \int_x^L q(x)dx + N(L)$$

From our boundary conditions, we know that $N(L) = 0$, and we thus get our solution:

$$N(x) = \int_x^L q(x)dx = \Big[q_0 \frac{x^2}{2L} + q_1 x\Big]_x^L = -q_0 \frac{x^2}{2L} - q_1 x + \frac{q_0 L}{2} + q_1 L$$

Note : this is equivalent to writing $u'(x) = \int \frac{-q(x)}{AE} dx + C$ without specifying the boundaries of the integral and determining the value of C with the boundary condition (as we did in the previous question)