

Exercise 1

The mechanical properties of a healed femur bone are being investigated. In the place where the break healed, a thickening of the bone has occurred. A simplified sketch of the structure can be seen in figure 1. The bone has a uniform Young's modulus of $E_1 = 60 \text{ GPa}$ in the unbroken part and $E_2 = 40 \text{ GPa} = \frac{2}{3}E_1$ in the healed part, a segment length of $L = 20 \text{ cm}$ and a cross sectional area of $A = 5 \text{ cm}^2$.

If the bone is clamped at both ends and stressed by a force $F = 11 \text{ kN}$ at the thickened part, calculate the external reaction forces and displacements. Use the displacement stiffness method.

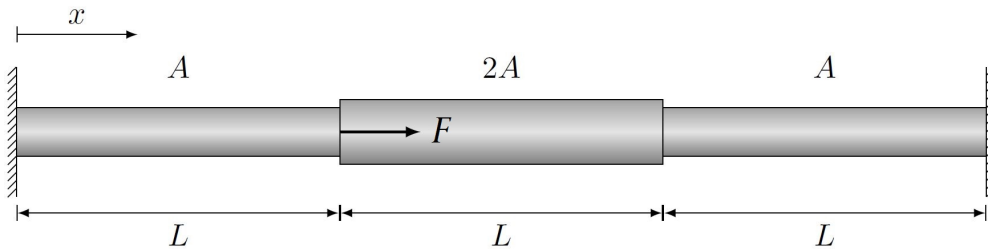


Figure 1: Simplified structure of a bone with a fortification in the middle that is being loaded.

Exercise solution 1

What is given:

- Young's Moduli $E_1 = 60 \text{ GPa}$ and $E_2 = 40 \text{ GPa} = \frac{2}{3}E_1$
- Cross-sectional area $A = 5 \text{ cm}^2$
- Segment length $L = 20 \text{ cm}$
- Loading force $F = 11 \text{ kN}$

What is asked:

- Displacements u_2, u_3
- Reaction forces F_1, F_4

Applicable formulas:

- Hooke's Law $E\varepsilon = \sigma$

1. As we have two boundaries and two discontinuities we find four nodes.

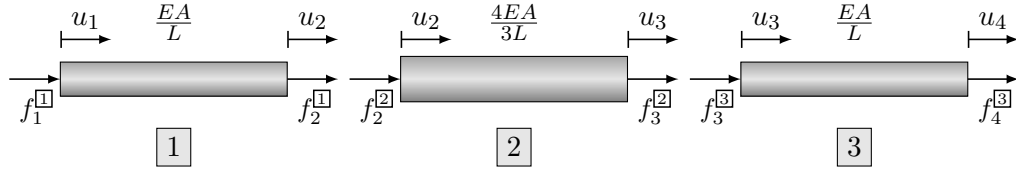


Figure 2: Sectioned system with local displacements and local forces indicated.

2. We cut the system into pieces at the discontinuities to get three individual segments that can be treated locally (see figure 2).
3. We find the local force stiffness relations of the system as

$$\begin{aligned} f_1^{[1]} &= k_1 \cdot (u_1 - u_2) & f_2^{[2]} &= k_2 \cdot (u_2 - u_3) & f_3^{[3]} &= k_3 \cdot (u_3 - u_4) \\ f_2^{[1]} &= k_1 \cdot (u_2 - u_1) & f_3^{[2]} &= k_2 \cdot (u_3 - u_2) & f_4^{[3]} &= k_3 \cdot (u_4 - u_3) \end{aligned}$$

which we can rewrite in matrix form as

$$\underbrace{\begin{Bmatrix} f_1^{[1]} \\ f_2^{[1]} \end{Bmatrix}}_{\mathbf{k}^{[1]}} = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \begin{Bmatrix} f_2^{[2]} \\ f_3^{[2]} \end{Bmatrix} = \mathbf{k}^{[2]} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \quad \begin{Bmatrix} f_3^{[3]} \\ f_4^{[3]} \end{Bmatrix} = \mathbf{k}^{[3]} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} \quad (1)$$

where the stiffness k is given by Hooke's law as

$$k_1 = \frac{E_1 A}{L} \quad k_2 = \frac{E \cdot \frac{4}{3} A}{L} = \frac{4}{3} \frac{E_1 A}{L} \quad k_3 = \frac{E_1 A}{L}$$

4. The formulas in (1) represent local stiffness relations which can be expanded to the global form by filling the matrices appropriately with zeroes

$$\begin{aligned} \begin{Bmatrix} f_1^{[1]} \\ f_2^{[1]} \\ 0 \\ 0 \end{Bmatrix} &= \frac{E_1 A}{L} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \\ \begin{Bmatrix} 0 \\ f_2^{[2]} \\ f_3^{[2]} \\ 0 \end{Bmatrix} &= \frac{4}{3} \frac{E_1 A}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \\ \begin{Bmatrix} 0 \\ 0 \\ f_3^{[3]} \\ f_4^{[3]} \end{Bmatrix} &= \frac{E_1 A}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \end{aligned}$$

and finally superimposing the three subproblems to get the full displacement stiffness equation for the system as

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{Bmatrix} f_1^{[1]} \\ f_2^{[1]} + f_2^{[2]} \\ f_3^{[2]} + f_3^{[3]} \\ f_4^{[3]} \end{Bmatrix} = \frac{E_1 A}{L} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & \frac{7}{3} & -\frac{4}{3} & 0 \\ 0 & -\frac{4}{3} & \frac{7}{3} & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \quad (2)$$

5. To solve this we use the boundary conditions that the system is clamped at both ends ($u_1 = u_4 = 0$) as well as that we know the external forces around the thicker part ($F_2 = F$, $F_3 = 0$ (no external force at the node 3!)) which results in the reduced system

$$\begin{Bmatrix} F \\ 0 \end{Bmatrix} = \frac{E_1 A}{L} \cdot \begin{bmatrix} \frac{7}{3} & -\frac{4}{3} \\ -\frac{4}{3} & \frac{7}{3} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

6. The reduced system can be solved either with substitution or by matrix inversion to get the nodal displacements

$$u_2 = \frac{7}{11} \frac{FL}{E_1 A}, \quad u_3 = \frac{4}{11} \frac{FL}{E_1 A}$$

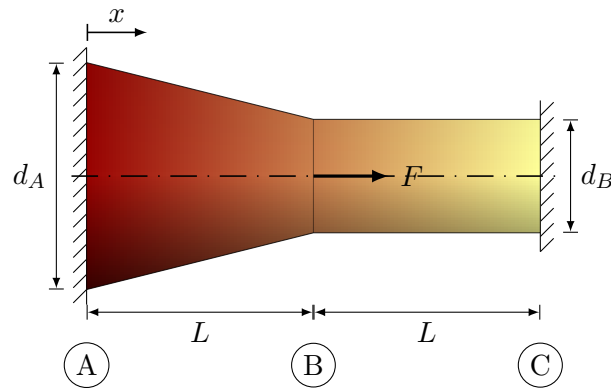
7. Putting this back into the matrix equation (2) we get the reaction forces $F_1 = -\frac{7}{11}F$ and $F_4 = -\frac{4}{11}F$.

Finally we calculate $u_2 = 46.6 \mu\text{m}$, $u_3 = 36.7 \mu\text{m}$, $F_1 = -7 \text{ kN}$ and $F_4 = -4 \text{ kN}$ as numerical values.

Exercise 2

A force F acts at point B on a structure (see Fig. 3). The structure is clamped on both sides and the maximum diameter of the structure is d_A while the small diameter is d_B . The structure has a Young's modulus E and a length $2L$.

- Define the nodes in the system. Deduce the displacement-stiffness equation for the system. You will use k_1 as the stiffness of the left part of the system (between points A and B) and k_2 as the stiffness of the right part of the system (between points B and C), without trying to find their value yet.
- For this question and the next one, we consider that $d_A = 2d$ and $d_B = d$. Determine k_1 and k_2 as a function of E , L and d .
- Deduce the reaction forces R_A and R_C , as well as the displacement u_B , as a function of E , L , d and F . Draw the reaction forces on the schematics and respect the sign convention.

Figure 3: Force F acting on the structure.

Exercise solution 2

Given:

Young's Modulus E

Diameters d_A and d_B

Segments lengths L

Force F at point B

Asked:

Displacement at point B

Reaction forces at points A and C

Relevant relationships:

Hooke's Law $E\varepsilon = \sigma$

Kinematic equation $\varepsilon(x) = \frac{\partial u}{\partial x}(x)$

Equilibrium equation $\frac{\partial N}{\partial x}(x) = 0$ when there are no distributed forces.

a)

First we cut the system into pieces at the discontinuities to get two individual segments that can be treated locally (see figure 4). There are 3 nodes : at A, B and C.

We find the local force stiffness relations of the system in the matrix form as

$$\begin{bmatrix} f_1^{[1]} \\ f_2^{[1]} \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} f_1^{[2]} \\ f_2^{[2]} \end{bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}$$

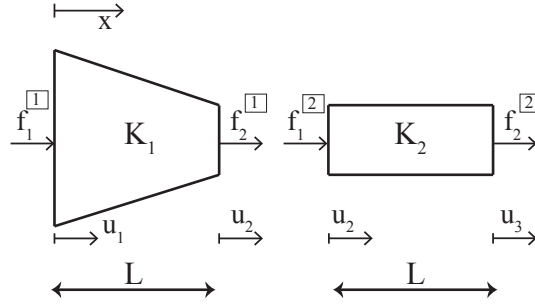


Figure 4: Sectioned system with local displacements and local forces indicated.

where k_1 and k_2 are the stiffnesses of the two sections in the system. Now we can get the full displacement stiffness equation for the system as

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} f_1^{[1]} \\ f_2^{[1]} + f_1^{[2]} \\ f_2^{[2]} \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

b)

We have to find k_1 and k_2 . We already know the formula for the stiffness of a bar :

$$k_2 = \frac{A_2 \cdot E}{L} = \frac{\pi d^2 E}{4L} = \frac{E \pi d^2}{4L}$$

k_1 needs to be calculated manually as it does not have a constant section area. By definition, $k_1 = \frac{N}{\Delta L_1} = \frac{N}{u_B - u_A}$ where N is the normal internal force within the object. Because there are no distributed loads, we can already notice that N is constant (cf. equilibrium equation) and does not depend on x .

Let's determine the elongation $\Delta L_1 = u_B - u_A$:

$$\begin{aligned} u_B - u_A &= \int_A^B \varepsilon(x) dx \\ &= \frac{1}{E} \int_A^B \sigma(x) dx && \text{Hooke's law} \\ &= \frac{N}{E} \int_A^B \frac{1}{A(x)} dx && \sigma(x) = N/A(x) \text{ and } N \text{ is constant} \\ &= \frac{N}{E\pi} \int_A^B r^{-2}(x) dx \end{aligned}$$

As a consequence,

$$k_1 = \frac{N}{u_B - u_A} = \frac{E\pi}{\int_A^B r^{-2}(x) dx}$$

We need to find the radius of the bar as a function of length (see figure 5).

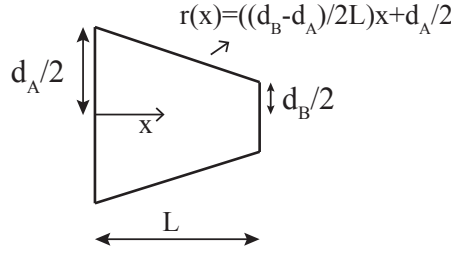


Figure 5: Radius of structure as a function of length.

Finally,

$$\begin{aligned}
 k_1 &= E\pi \frac{1}{\int_0^L \left(\frac{d}{2L}x + \frac{d}{2}\right)^{-2} dx} \\
 &= \frac{E\pi d^2}{4} \frac{1}{\int_0^L \left(1 + \frac{x}{L}\right)^{-2} dx} \\
 &= \frac{E\pi d^2}{4} \frac{1}{[-L(1 + x/L)^{-1}]_0^L} \\
 k_1 &= \frac{E\pi d^2}{2L}
 \end{aligned}$$

c)

Now we can put k_1 and k_2 in the full stiffness matrix. We also use the boundary conditions that the system is clamped at both sides $u_1 = u_3 = 0$ as well as that we know the force in point B ($F_2 = F$).

$$\begin{bmatrix} R_A \\ F \\ R_C \end{bmatrix} = \frac{E\pi d^2}{4L} \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix}$$

d)

Now we can calculate displacement u_2 and reaction forces R_A and R_C as:

$$\begin{aligned}
 u_2 &= u_B = \frac{4LF}{3\pi E d^2} \\
 R_A &= -\frac{\pi E d^2}{2L} \cdot u_2 = -\frac{\pi E d^2}{2L} \cdot \frac{4LF}{3\pi E d^2} = -\frac{2}{3}F \\
 R_C &= -\frac{\pi E d^2}{4L} \cdot u_2 = -\frac{\pi E d^2}{4L} \cdot \frac{4LF}{3\pi E d^2} = -\frac{1}{3}F
 \end{aligned}$$

And the sum of the external forces applied to the system R_A , R_C and F is equal to zero as the system is at equilibrium.

Exercise 3

A prosthesis part made out of an aluminum alloy has the geometry as depicted in figure 6. The widths are $b = 60$ mm, $c = 30$ mm, $d = 5$ mm and the thickness is $t = 5$ mm. The fillets have a radius of $R = 8$ mm. The ultimate strength of the aluminum alloy is $\sigma_U = 410$ MPa. The fillets and the hole are spaced by 80 mm.

For both questions, you can use figures 7 and 8 to determine the stress-concentration factor K .

- If a factor of safety of $SF=2.5$ is required, what is the maximum allowable tensile load P_a ?
- (*difficult question, won't be asked at the exam*) Find the hole diameter $d_{\max}$ at which the two segments of the bar have the same tensile load carrying capacity.

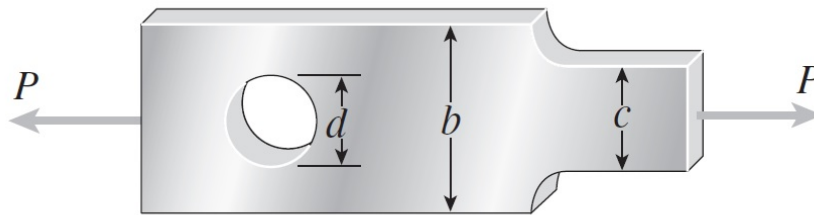


Figure 6: A prosthesis part with a hole.

Exercise solution 3

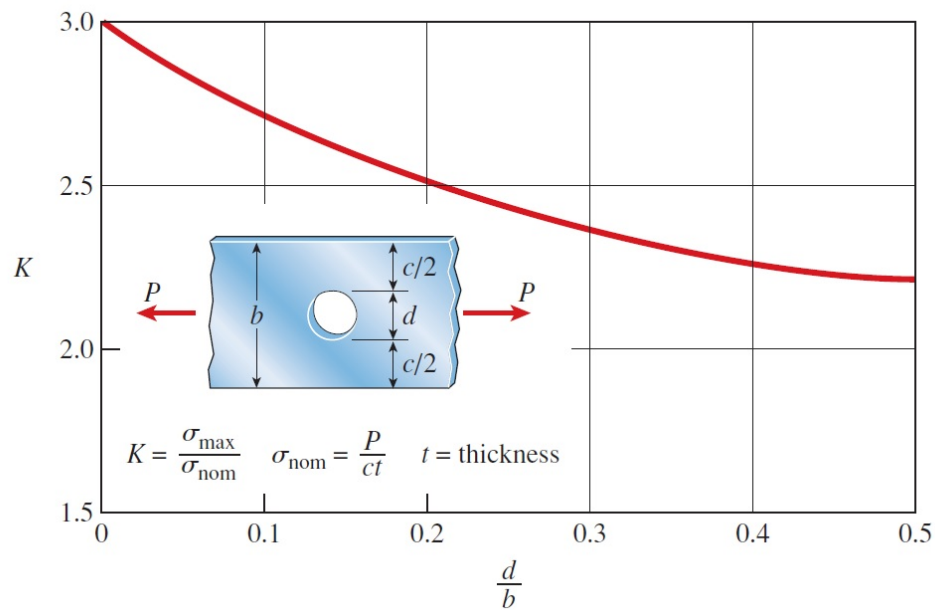
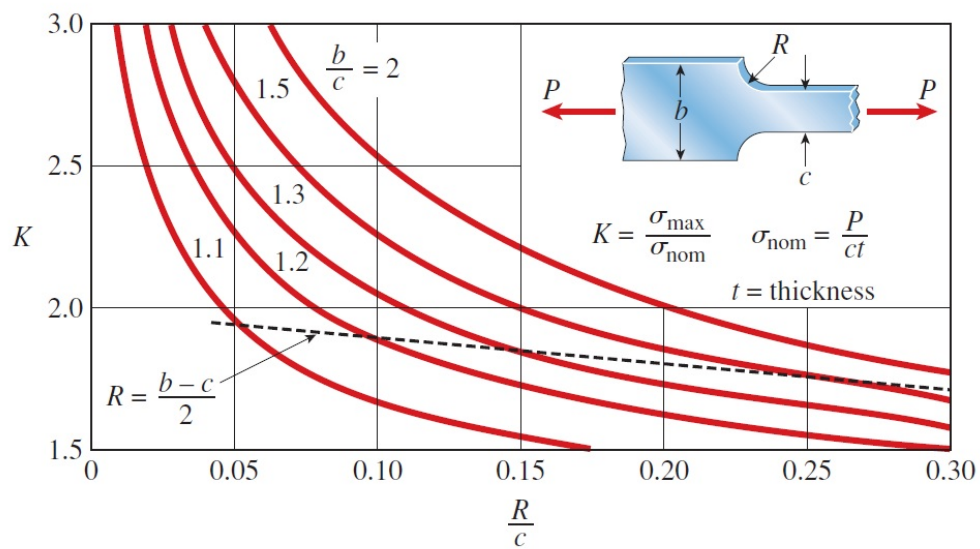
What is given:

- Force P
- Dimensions of parts
- Graphs of stress-concentration factors K

What is asked:

- Maximum tensile load $P_{\max}$
- Maximum hole diameter $d_{\max}$

Applicable formulas:

Figure 7: Stress-concentration factor K for flat bars with a circular hole.Figure 8: Stress-concentration factor K for flat bars with shoulder fillets.

- Stress concentration factor:

$$K = \frac{\sigma_{\max}}{\sigma_{\text{nom}}}$$

- Normal stress:

$$\sigma_{\text{nom}} = \frac{P}{ct}$$

a)

First, let's determine the value of the maximum stress allowed in the structure. It is given by the ultimate strength of the material and the safety factor :

$$\sigma_{\max} = \frac{\sigma_U}{SF}$$

We divide now the prosthesis part into two segments, the left segment containing the hole, and the right segment with the fillets. The two segments can be considered independently according to the Saint-Venant principle, as they are spaced by 80mm , which is more than the size of the hole / fillet area. Next, we need to calculate, for each segment, the force P which would create a stress $\sigma_{\max}$ in that particular segment.

Left segment The stress-concentration factor links the actual maximum stress in the structure $\sigma_{\max}$ and the stress calculated assuming no stress concentration σ_{nom} : $\sigma_{\max} = K_1 \cdot \sigma_{\text{nom}}$. The stress concentration factor is read out from the graph : $d/b = 5/60 = 0.08 \rightarrow K_1 = 2.8$.

Let's determine the nominal stress. It is the ratio between the force and the smallest area on which the force is applied : $A = (b - d)t$. So $\sigma_{\text{nom}} = \frac{P}{(b-d)t}$ (i.e. the maximum stress we would have if there was no stress concentration).

Finally, the maximum stress in the left segment is

$$\sigma_{\max} = K_1 \frac{P}{(b-d)t} = \frac{\sigma_U}{SF}$$

As a consequence, the maximum force that can be applied to the left segment is

$$\begin{aligned} P_1 &= \frac{\sigma_U}{SF} \frac{(b-d)t}{K_1} \\ &= \frac{410 \cdot 10^6}{2.5} \frac{(60-5) \cdot 5 \cdot 10^{-6}}{2.8} \\ &= 16.1\text{kN} \end{aligned}$$

Right segment It is the same method for the right segment : $R/c = 8/30 = 0.27$ and $b/c = 60/30 = 2$ so $K_2 = 1.8$ according to the graph for stress concentration. $\sigma_{\text{nom}} = P/(ct)$ and, finally,

$$\begin{aligned} P_2 &= \frac{\sigma_U}{SF} \frac{ct}{K_2} \\ &= \frac{410 \cdot 10^6}{2.5} \frac{30 \cdot 5 \cdot 10^{-6}}{1.8} \\ &= 13.7\text{kN} \end{aligned}$$

Maximal allowable tensile force The left segment is reaching its limit for $P_1 = 16.1kN$ and the right segment for $P_2 = 13.7kN$. P_A has to be lower or equal to both the limits of the left and the right segment. As a consequence, $P_A = \min(P_1, P_2) = P_2 = 13.7kN$.

b)

We want to determine the value of the diameter d of the hole in order to have $P_A = P_1 = P_2 = 13.7kN$. It means we can increase the diameter of the hole. The diameter of the hole in the left segment of the prosthesis bar has an impact on two values for the left segment :

- the nominal stress : $\sigma_{nom} = \frac{P}{(b-d)t}$
- the stress concentration factor : K determined on the graph as a function of d/b

Let's express K_1 as a function of d/b .

$$\begin{aligned}
 P_1 &= P_2 \\
 \frac{\sigma_U}{SF} \frac{(b-d)t}{K_1} &= P_2 \\
 K_1 &= \frac{\sigma_U b t}{SF \cdot P_2} \left(1 - \frac{d}{b}\right) \\
 K_1 &= \frac{410 \cdot 10^6 \cdot 60 \cdot 10^{-3} \cdot 5 \cdot 10^{-3}}{2.5 \cdot 13.7 \cdot 10^3} \cdot \left(1 - \frac{d}{b}\right) \\
 K_1 &= 3.59 \cdot \left(1 - \frac{d}{b}\right)
 \end{aligned}$$

The value of K_1 will be at the intersection of the red curve $K_1 = f(d/b)$ and the blue line defined by the equation $K_1 = 3.59 \cdot (1 - \frac{d}{b})$, as described figure 9. Finally, $d/b = 0.35$, so the diameter of the hole is $d_{max} = 0.35 \cdot 60 = 21mm$. Let's check our calculation. We now have

$$\begin{aligned}
 P_1 &= \frac{\sigma_U}{SF} \frac{(b-d)t}{K_1} \\
 &= \frac{410 \cdot 10^6}{2.5} \frac{(60 - 21) \cdot 5 \cdot 10^{-6}}{2.3} \\
 &= 13.9kN \\
 &\approx 13.7kN = P_2
 \end{aligned}$$

OK given the measurement approximations

and the two elements effectively have the same tensile load carrying capacity for $d_{max} = 21mm$.

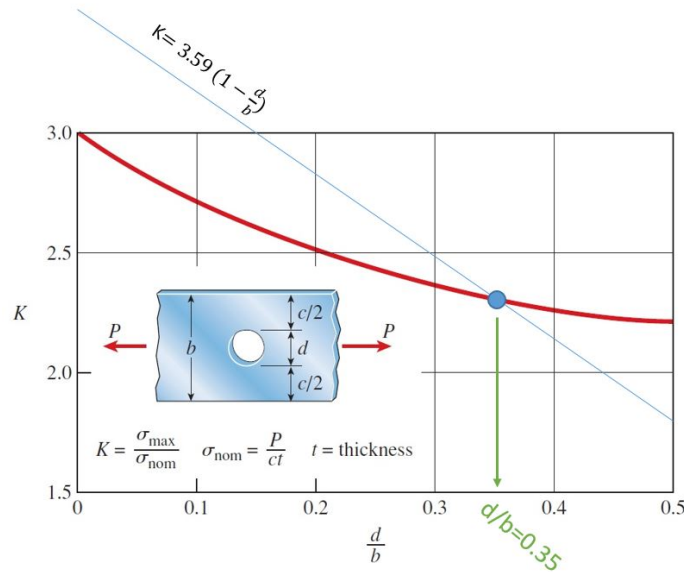


Figure 9: Determination of the value of d/b to get a given stress in the left segment

Exercise 4

We consider heated bars in different configurations.

- a) A simple aluminum bar (Young's Modulus $E_{Al} = 69\text{GPa}$, linear temperature expansion coefficient $\alpha_{Al} = 24 \cdot 10^{-6}\text{K}^{-1}$) with a square-cross section is clamped at one end (fig. 10a). At room temperature $T_{room} = 22^\circ\text{C}$, its length is $L = 30\text{cm}$ and its width $w = 5\text{cm}$.

Calculate the elongation of the bar if the air conditioning fails and the temperature increases to $T_{summer} = 37^\circ\text{C}$.

- b) The bar is back at room temperature $T_{room} = 22^\circ\text{C}$. A force P is applied at the open end (fig. 10b).

Give an expression for the total shortening of the bar as a function of P .

- c) The bar is now clamped on both sides.

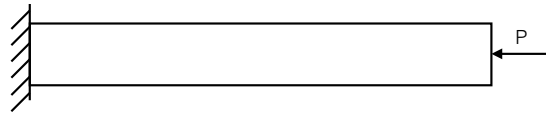
Calculate the stress in the bar and the reaction forces at both ends when the temperature increases to $T_{summer} = 37^\circ\text{C}$.

- d) The aluminum bar is replaced by a bi-material model. While the external dimensions are the same, it consists of two half-bars of length $L/2$, one of aluminum and the other of copper ($E_{Cu} = 117\text{GPa}$, $\alpha_{Cu} = 16 \cdot 10^{-6}\text{K}^{-1}$).

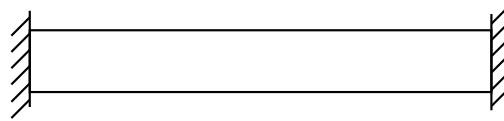
Find the displacement of the aluminum-copper intersection as well as the stress in the bar when the ambient temperature increases by $\Delta T = 15^\circ\text{C}$.



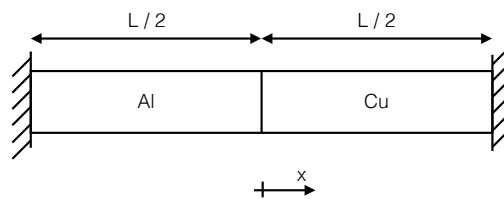
(a) Dimensions of the aluminum bar.



(b) A compressive force P is applied at the open end.



(c) The bar is clamped on both sides.



(d) Dimensions of the Aluminum-Copper bar.

Figure 10: Schematics of the bars.

Exercise solution 4

Given:

Temperature change $\Delta T = 15K$

Dimensions of the bar $L = 30cm$ and $w = 5cm$

Thermal coefficients of aluminum $\alpha_{Al} = 24 \cdot 10^{-6} K^{-1}$ and copper $\alpha_{Cu} = 16 \cdot 10^{-6} K^{-1}$

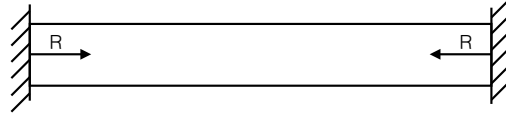
Young's modulus of aluminum $E_{Al} = 69GPa$ and copper $E_{Cu} = 117GPa$

Asked: Total elongation (a), total shortening as a function of P (b), stress and reaction forces (c), stress and displacement of the intersection (d).

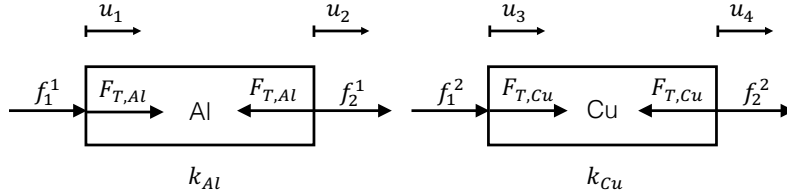
Relevant relationships:

Thermal strain : $\varepsilon_T = \alpha \Delta T$

Hooke's law : $\sigma = E\varepsilon$



(a) The bar's thermal expansion is compensated by the reaction forces.



(b) Elements and nodes for the displacement-stiffness method.

Figure 11: Schematics of the bars.

Extended displacement stiffness matrix equation : $\vec{F} = \mathbf{k} \cdot \vec{u} + \vec{q}$

- a) This is a simple application of the thermal expansion of materials. We are interested in the total elongation of the bar in the x-direction:

$$\Delta L_T = \varepsilon_T \cdot L = \Delta T \cdot \alpha_{Al} \cdot L$$

$$\Delta L_T = 15[K] \cdot 24 \cdot 10^{-6}[K^{-1}] \cdot 0.3[m] = 0.11 \cdot 10^{-3}m$$

- b) Here we can directly apply Hooke's law to find the total shortening:

$$\Delta L_P = \frac{P}{k_{bar}} = \frac{P \cdot L}{A \cdot E} = \frac{P \cdot L}{w^2 \cdot E_{Al}}$$

$$\Delta L_P = \frac{P \cdot 30 \cdot 10^{-2}[m]}{25 \cdot 10^{-4}[m^2] \cdot 69 \cdot 10^9[N/m^2]} = P \cdot 1.79 \cdot 10^{-9}m$$

- c) The bar is now clamped at both ends, then the temperature increases. We will have two effects: the bar will elongate due to the thermal expansion (ΔL_T), but since the bar is constrained, the reaction forces at both ends will shorten (ΔL_P) the bar to its initial length.

We could use a displacement stiffness method to solve this, but we can also simply use the results from parts (a) and (b). The total elongation must be zero, and we will rename the force P as R (fig. 11a):

$$\Delta L_T - \Delta L_R = 0 \rightarrow \Delta L_T = \Delta T \cdot \alpha \cdot L = \Delta L_R = \frac{R \cdot L}{w^2 \cdot E_{Al}}$$

$$R = \Delta T \cdot \alpha_{Al} \cdot w^2 \cdot E_{Al}$$

$$R = 15[K] \cdot 24 \cdot 10^{-6}[K^{-1}] \cdot 25 \cdot 10^{-4}[m^2] \cdot 69 \cdot 10^9[N/m^2] = 62.1 kN$$

The stress in the bar is thus given by:

$$\sigma_{Al} = -\frac{R}{w^2} = -\frac{62.1 \cdot 10^3[N]}{25 \cdot 10^{-4}[m^2]} = -24.8 MPa$$

d) To solve this part, we have two approaches.

Approach 1

Probably the more intuitive approach. Similarly to what we did in point (c) above, we will first calculate the length that the bar would have after the temperature change, if it were not constrained. This will be a new initial length L^* (after temperature change). Since in reality the bar is constrained, it cannot be longer than L . We can now use the displacement-stiffness method (fig. 11b), choosing the boundary condition at the right as $u_4 = L - L^*$. In other words the reaction force must be such as to induce a displacement u_4 that will exactly compensate the thermal expansion of the bar.

Boundary conditions First we calculate L^* , the length each bar would have if not constrained:

$$L_{Al}^* = \frac{L}{2}(1 + \alpha_{Al}\Delta T)$$

$$L_{Cu}^* = \frac{L}{2}(1 + \alpha_{Cu}\Delta T)$$

We can find our boundary condition for the displacement stiffness method:

$$\begin{aligned} u_4 &= L - (L_{Al}^* + L_{Cu}^*) = L - \frac{L}{2}(1 + \alpha_{Al}\Delta T) - \frac{L}{2}(1 + \alpha_{Cu}\Delta T) \\ &= -\frac{L}{2}\Delta T(\alpha_{Al} + \alpha_{Cu}) \end{aligned}$$

Displacement-stiffness matrix We now use the normal displacement-stiffness matrix formula $\vec{F} = \mathbf{k} \cdot \vec{u}$. Knowing that $u_2 = u_3$, we obtain:

$$\begin{aligned} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} &= \begin{bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 \end{bmatrix} \\ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} &= \begin{bmatrix} k_{Al} & -k_{Al} & 0 \\ -k_{Al} & k_{Al} + k_{Cu} & -k_{Cu} \\ 0 & -k_{Cu} & k_{Cu} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix} \end{aligned}$$

Boundary conditions The system is clamped on both sides : $u_1 = 0$ and $u_4 = L - L^*$. No external force acts at the aluminum-copper intersection, therefore $F_2 = 0$. F_1 and F_3 are the reaction forces of the clamping.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 \\ L - L^* \end{bmatrix} \qquad \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} R_{Al} \\ 0 \\ R_{Cu} \end{bmatrix}$$

Solution To solve the system, we insert the boundary conditions into the previous equation, which becomes :

$$\begin{bmatrix} R_{Al} \\ 0 \\ R_{Cu} \end{bmatrix} = \frac{2A}{L} \begin{bmatrix} E_{Al} & -E_{Al} & 0 \\ -E_{Al} & E_{Al} + E_{Cu} & -E_{Cu} \\ 0 & -E_{Cu} & E_{Cu} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ u_2 \\ -\frac{L}{2}\Delta T(\alpha_{Al} + \alpha_{Cu}) \end{bmatrix}$$

From the second line, one can determine the value of u_2 :

$$\begin{aligned} 0 &= \frac{2A}{L} (E_{Al} + E_{Cu}) \cdot u_2 + \Delta T \cdot A \cdot E_{Cu} \cdot (\alpha_{Al} + \alpha_{Cu}) \\ u_2 &= -\Delta T \cdot \frac{L}{2} \cdot (\alpha_{Al} + \alpha_{Cu}) \frac{E_{Cu}}{E_{Al} + E_{Cu}} \\ u_2 &= -56.6 \mu m \end{aligned}$$

The displacement of the intersection is thus given by:

$$d_2 = \frac{L}{2} \cdot \Delta T \cdot \alpha_{Al} + u_2 = 54 - 56.6 = -2.6 \mu m$$

To find the stress in the bar, we need to find the reaction forces of the clamping. From the first line, if we put back the value of u_2 into the equation, we get R_{Al} and thus the stress σ :

$$\begin{aligned} R_{Al} &= -\frac{2A}{L} \cdot E_{Al} \cdot u_2 = A \cdot \Delta T \cdot (\alpha_{Al} + \alpha_{Cu}) \frac{E_{Al} \cdot E_{Cu}}{E_{Al} + E_{Cu}} \\ \sigma &= -\frac{R_{Al}}{A} = -\Delta T \frac{E_{Al} \cdot E_{Cu}}{E_{Al} + E_{Cu}} (\alpha_{Al} + \alpha_{Cu}) \\ \sigma &= -26.0 MPa \end{aligned}$$

Approach 2

Alternatively we can directly use the displacement-stiffness method, with an added term for the thermal forces. Note that these thermal forces only

exist if an element is constrained; indeed a free element that experiences a change of temperature will be strained but not stressed! The forces are drawn in fig. 11b. First, we have our local stiffness matrixes:

$$\begin{bmatrix} f_1^1 \\ f_2^1 \end{bmatrix} = \frac{A_{Al}E_{Al}}{L/2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} f_1^2 \\ f_2^2 \end{bmatrix} = \frac{A_{Cu}E_{Cu}}{L/2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_3 \\ u_4 \end{bmatrix}$$

We need to add in the thermal forces. To determine them, we assume that the bar experiences thermal strain but is constrained:

$$F_T = A \cdot \sigma_T = A \cdot E \varepsilon_T = A \cdot E \cdot \alpha \cdot \Delta T$$

$$F_{T,Al} = A_{Al} \cdot E_{Al} \cdot \alpha_{Al} \cdot \Delta T$$

$$F_{T,Cu} = A_{Cu} \cdot E_{Cu} \cdot \alpha_{Cu} \cdot \Delta T$$

Displacement-stiffness matrix Instead of the normal displacement-stiffness matrix formula $\vec{F} = \mathbf{k} \cdot \vec{u}$, we use the extended equation $\vec{F} = \mathbf{k} \cdot \vec{u} + \vec{q}$ where $\vec{q}$ is a vector containing the thermal forces. Determining the signs according to fig. 11b and knowing that $u_2 = u_3$, we obtain:

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 \end{bmatrix}$$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} k_{Al} & -k_{Al} & 0 \\ -k_{Al} & k_{Al} + k_{Cu} & -k_{Cu} \\ 0 & -k_{Cu} & k_{Cu} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix} + \begin{bmatrix} F_{T,Al} \\ -F_{T,Al} + F_{T,Cu} \\ -F_{T,Cu} \end{bmatrix}$$

Boundary conditions The system is clamped on both sides : $u_1 = u_4 = 0$. No external force acts at the aluminum-copper intersection, therefore $F_2 = 0$. F_1 and F_3 are the reaction forces of the clamping.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} R_{Al} \\ 0 \\ R_{Cu} \end{bmatrix}$$

Solution To solve the system, we insert the boundary conditions into the previous equation, which becomes :

$$\begin{bmatrix} R_{Al} \\ 0 \\ R_{Cu} \end{bmatrix} = \frac{2A}{L} \begin{bmatrix} E_{Al} & -E_{Al} & 0 \\ -E_{Al} & E_{Al} + E_{Cu} & -E_{Cu} \\ 0 & -E_{Cu} & E_{Cu} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix} + \Delta T \cdot A \cdot \begin{bmatrix} E_{Al} \alpha_{Al} \\ -E_{Al} \alpha_{Al} + E_{Cu} \alpha_{Cu} \\ -E_{Cu} \alpha_{Cu} \end{bmatrix}$$

From the second line, one can determine the value of u_2 which is equivalent to the displacement of the intersection:

$$\begin{aligned} 0 &= \frac{2A}{L} (E_{Al} + E_{Cu}) \cdot u_2 + \Delta T \cdot A \cdot (-E_{Al} \alpha_{Al} + E_{Cu} \alpha_{Cu}) \\ u_2 &= \Delta T \cdot \frac{L}{2} \cdot \frac{E_{Al} \alpha_{Al} - E_{Cu} \alpha_{Cu}}{E_{Al} + E_{Cu}} \\ d_2 = u_2 &= -2.6 \mu m \end{aligned}$$

To find the stress in the bar, we need to find the reaction forces of the clamping. From the first line, if we put back the value of u_2 into the equation, we get R_{Al} and thus the stress σ :

$$\begin{aligned} R_{Al} &= \frac{2A}{L} (-E_{Al} \cdot \Delta T \cdot \frac{L}{2} \frac{E_{Al} \alpha_{Al} - E_{Cu} \alpha_{Cu}}{E_{Al} + E_{Cu}}) + \Delta T A E_{Al} \alpha_{Al} \\ R_{Al} &= A \Delta T (E_{Al} \alpha_{Al} - E_{Al} \frac{E_{Al} \alpha_{Al} - E_{Cu} \alpha_{Cu}}{E_{Al} + E_{Cu}}) \\ R_{Al} &= A \Delta T \frac{E_{Al} E_{Cu}}{E_{Al} + E_{Cu}} (\alpha_{Al} + \alpha_{Cu}) \\ \sigma &= -\frac{R_{Al}}{A} = -\Delta T \frac{E_{Al} E_{Cu}}{E_{Al} + E_{Cu}} (\alpha_{Al} + \alpha_{Cu}) \\ \sigma &= -26.0 MPa \end{aligned}$$