Exercise 1

Goals of the exercise: (1) understand the difference between macroscopic and microscopic strain, (2) understand qualitatively the relationship between the local deformation $u(x)$ and the local strain $\varepsilon(x)$.

a) We consider the bar in figure 1 before and after applying a force in the longitudinal axis. Use a ruler to measure the distances on the figure and answer the following questions.

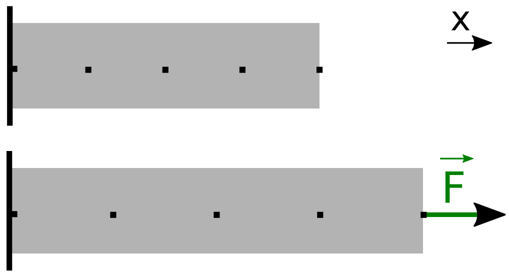

Figure 1: Longitudinal deformation of a rectangular bar

1. What is the value of the macroscopic strain ?
2. Represent on the figure the vector $\vec{u}$ for the third drawn point. What is the displacement $u(x)$ of each of the drawn points ? Plot $u(x)$ on a graph (you can do a linear interpolation between the points).
3. Determine the value of the strain in each individual block (between two points) using the definition of strain ($\varepsilon = \frac{\Delta L}{L}$) and plot it. Show that you get the same answer as with the formula $\varepsilon(x) = \frac{du}{dx}(x)$ demonstrated during the class.
4. What is the relationship between the macroscopic strain and the microscopic strain ?

b) Same questions for the object in figure 2. Conclusion : what are the links and differences between ε, u, the macroscopic strain and the object's total elongation?

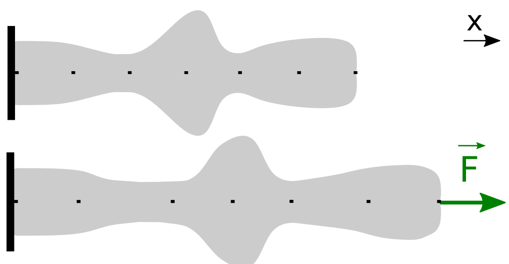

Figure 2: Longitudinal deformation of a free form object

Exercise solution 1

Warning : for the measurement of the distances, you might not get the exact same numerical values as in the solution depending on your printing parameters. But the values you get must be proportional to the ones given in the solution.

Given: Pictures of an object before and after applying a force.

Asked: Macroscopic strain, microscopic strain, local and total deformation; relationships between them
Note : for the measurement of the distances, you might not get the exact same numerical values as in the solution depending on your printing parameters. However the values you get must be proportional to the ones given in the solution.

a)

The macroscopic strain is defined by $\varepsilon_{macro} = \frac{L-L_0}{L_0} = \frac{48mm-36mm}{36mm} = 0.33$.
The positions of every point are measured with a ruler on the image and given in the following tab. An example of a displacement vector is given in figure 3. The local deformation u is easily deduced from the position values as $u_i = x_{i,final} - x_{i,initial}$.

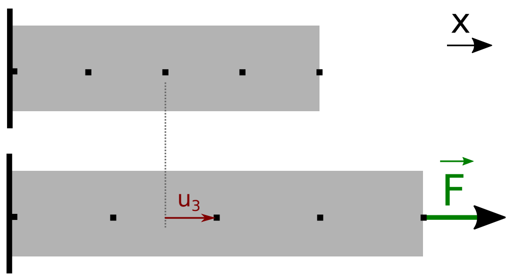

Figure 3: Displacement of the third point

point number	1	2	3	4	5
initial position (mm)	0	9	18	27	36
final position (mm)	0	12	24	36	48
u (mm)	0	3	6	9	12

We now want to determine the value of the strain in each individual blocks between two consecutive points. By definition, the strain of the block i between the points i and $i+1$ is $\varepsilon_i = \frac{L_{i,final}-L_{i,initial}}{L_{i,initial}}$.

block number	1	2	3	4
initial length of the block (mm)	9	9	9	9
final length of the block (mm)	12	12	12	12
strain ε	0.33	0.33	0.33	0.33

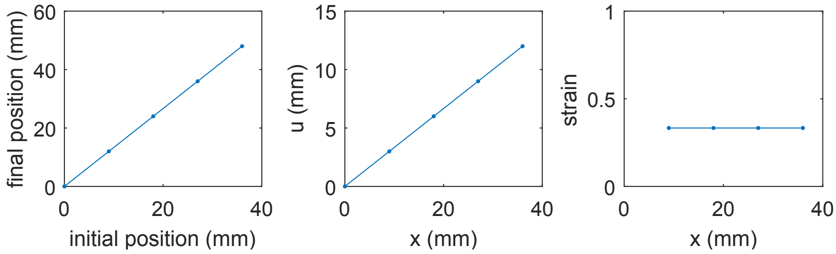

Figure 4: Displacement and strain within the bar

We see that for a homogeneous bar of constant section, the strain is uniform in the bar. It is also the same value as the macroscopic strain in the whole bar. It is something you should be able to prove in the general case with the method of sections.
During the class, you proved that $\varepsilon(x) = \frac{du}{dx}(x)$. Let's see if we find the same values with this formula. We just need to use, for dx, the distance between two consecutive points, and for du, the difference between two consecutive values of u given in the previous tab.

point number	2	3	4	5
dx **(mm)**	9	9	9	9
du **(mm)**	3	3	3	3
strain ε	0.33	0.33	0.33	0.33

We find the same value of strain (0.33) as previously.

b)

We use the same method as in the previous question to measure the position of the points, their displacement, and deduce the strain in the object.

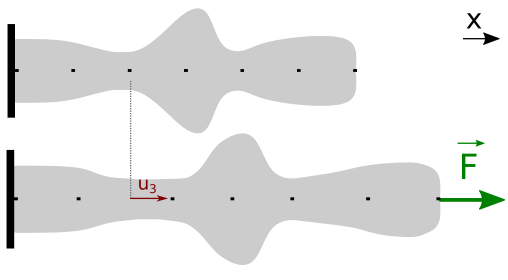

Figure 5: Displacement of the third point

point number	1	2	3	4	5	6	7
initial position (mm)	0	7	14	20	26	33	40
final position (mm)	0	8	18	26	33	42	50
u (mm)	0	1	4	6	7	9	10
ε		0.14	0.42	0.33	0.17	0.29	0.14

This time, the object is no longer a homogeneous bar with a constant section. Therefore, some parts of the bar (the thinner parts) elongate more than the thicker ones. The strain is therefore higher in these areas. The macroscopic strain is $\varepsilon_{macro} = \frac{50-40}{40} = 0.25$, which is not equal to the strain in every point of the bar. The macroscopic strain describes the behavior of the whole object, whereas the microscopic strain describes the strain in every point of the object.

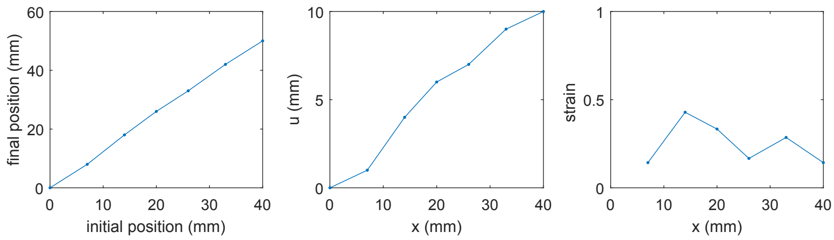

Figure 6: Displacement and strain within the object

Exercise 2

The crane structure shown in figure 7 is built with individual bars on rotating hinges (white dots). Vertical and horizontal bars are of length L, diagonal ones of length $\sqrt{2}L$. The crane is supported in point A with a fixed hinge (supports forces in x and in y direction) and in point B with a sliding hinge (supports forces in y direction only). The force $\vec{F}_C = (1, -3) \cdot F_c$ is acting on the point C. We will neglect the weight of the crane.

a) Cut the system free (replace the hinges in A and B with replacement forces).

b) Calculate the reaction forces in A and B as function of F_C.

c) Calculate the internal reaction forces in the beams 2–4, 12 and 13.

Exercise solution 2

Given: Truss structure, Force $\vec{F}_C$.

Asked: a) Replacement reaction forces for the base hinges. b) Values of the reaction forces at the base. c) Internal forces acting in the beams 12,13 and 2–4.

a)

See figure 8.

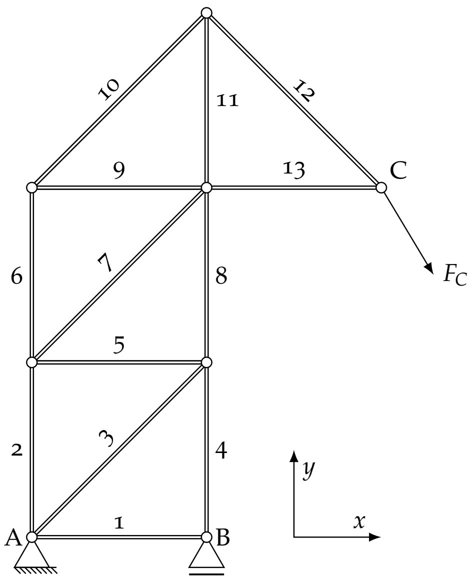

Figure 7: Crane structure with individual bars on rotating hinges .

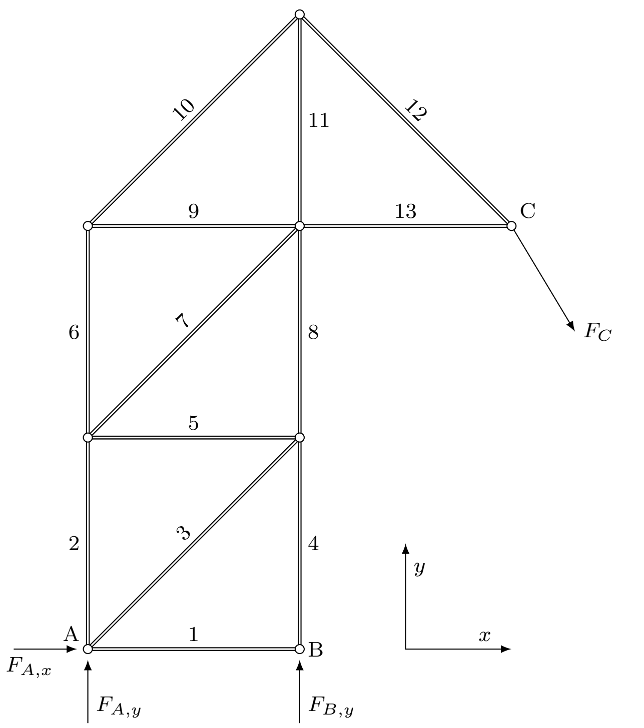

Figure 8: Crane with reaction forces.

b)

Reaction forces can be obtained with equilibrium of forces and moments.

$$\sum F_x = 0 \quad \rightarrow \quad F_{A,x} + F_{C,x} = 0 \quad \rightarrow \quad F_{A,x} = -F_{C,x}$$

The rest of the forces is determined by equilibrium in y direction together with the equilibrium of moments in A. The easiest way to calculate the moment is using the vector version (cross product)

$$\vec{M} = \vec{r} \times \vec{F}$$

and since only the z component is used, simplifies to

$$M_z = r_x \cdot F_y - r_y \cdot F_x$$

so the system of equations becomes

$$\sum F_y = 0 \quad \rightarrow \quad F_{A,y} + F_{B,y} - 3F_C = 0$$
$$\sum M_z = 0 \quad \rightarrow \quad \cancel{L} \cdot F_B, y + 2\cancel{L} \cdot (-3F_C) - 2\cancel{L} \cdot F_C = 0$$

which is solves to

$$F_{B,y} = 8F_C \qquad F_{A,y} = -5F_C$$

c)

By cutting the structure apart, making sure to cut through the beams of interest, we can obtain the inner forces with equilibriums of forces and moments from the external forces. For the forces inside the bars 12 and 13 the equilibrium of

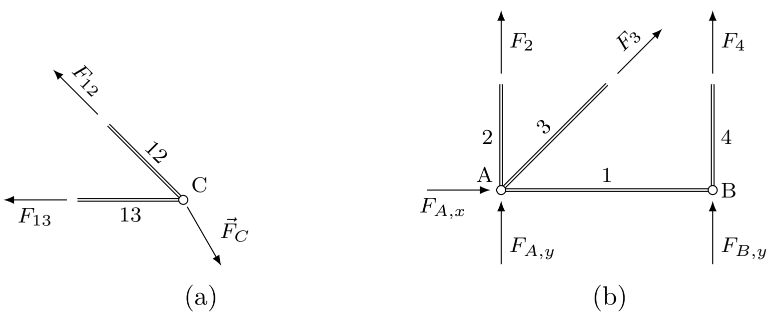

Figure 9: Virtual cuts at the nose and at the base

forces is sufficient:

$$\sum F_x = 0 \quad \rightarrow \quad -F_{13} - \overbrace{F_{12,x}}^{F_{12}/\sqrt{2}} + F_C = 0$$
$$\sum F_y = 0 \quad \rightarrow \quad -3F_C + \underbrace{F_{12,y}}_{F_{12}/\sqrt{2}} = 0$$
$$\rightarrow \quad F_{12} = 3\sqrt{2}F_C \qquad F_{13} = -2F_C$$

For the base we need both equilibriums of forces and moments (in A):

$$\sum F_x = 0 \quad \rightarrow \quad F_{A_x} + \frac{F_3}{\sqrt{2}} = 0 \quad \rightarrow \quad F_3 = \sqrt{2} F_C$$

$$\sum F_y = 0 \quad \rightarrow \quad F_2 + F_4 + F_{A,y} + F_{B,y} + \frac{F_3}{\sqrt{2}} = 0$$

$$\sum M_z = 0 \quad \rightarrow \quad \cancel{L} \cdot F_{B,y} + \cancel{L} \cdot F_4 = 0$$

$$\rightarrow \quad F_2 = -F_{A,y} - \frac{F_3}{\sqrt{2}} = 4F_C \quad F_4 = -8F_C$$

Exercise 3

A force $P = 1\,\mathrm{kN}$ is applied on a human femur bone (see figure 10(a)). The bone is modeled as a hollow tube with circular cross section and a constant wall thickness of 0.5 cm. The shape of the shaft of the bone is approximated by the quadratic function

$$y = \frac{x^2}{100\,\mathrm{cm}} + 2\,\mathrm{cm}$$

where the origin of x is in the middle of the bone (see figure 10(b)).

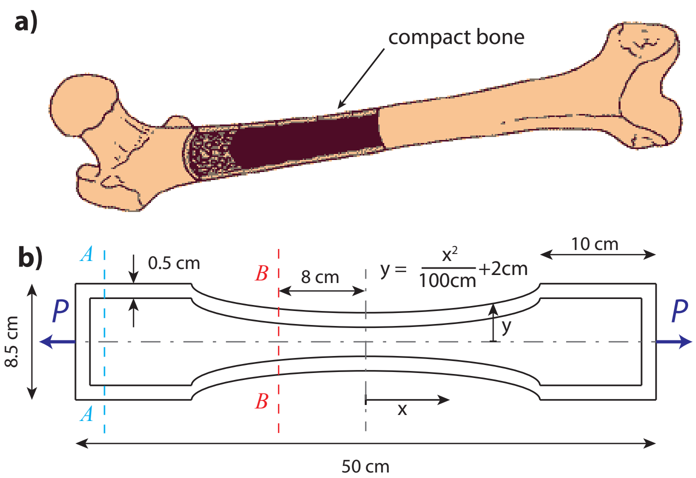

Figure 10: Illustration of human femur bone. a) Sketch of real bone, showing the compact bone wall. b) Sketch of the simplified model of the bone.

For the simplified model, find:

a) The stress in the cross–section A.

b) The stress in the cross–section B.

c) Where is the highest stress in the bone and what is its value ? If the load is increased, at which position will the bone break ?

Exercise solution 3

Given: Geometry, force $P = 1\,\text{kN}$ acting onto the bone.

Asked: Stresses in cross–sections A and B, point of failure and stress at that point.

Relevant relationships:

Definition of stress

$$\sigma = \frac{P}{A}$$

Area of a circle

$$A = \pi r^2$$

a) The stress on any part of the femur can be expressed by the formula

$$\sigma = \frac{P}{A} = \frac{P}{\pi \cdot (r_{\text{out}}^2 - r_{\text{in}}^2)} = \frac{P}{\pi t \cdot (2r_{\text{out}} - t)}$$

where t is the wall thickness of the compact bone.

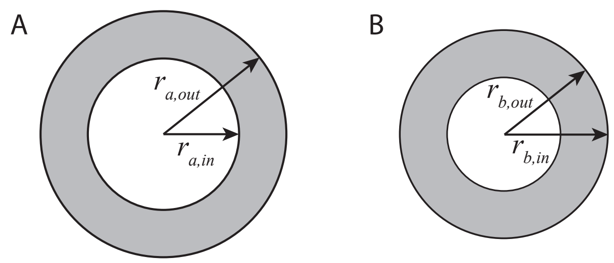

Figure 11: Ring cross-sections A and B.

This gives us for the cross-section A

$$\sigma_a = \frac{1\,\text{kN}}{\pi \cdot 0.5\,\text{cm} \cdot (8.5\,\text{cm} - 0.5\,\text{cm})} = 0.8\,\text{MPa}$$

b) The second cross–section is similar, except that we first have to calculate the outer radius

$$r_{b,\text{out}} = \frac{x^2}{100\,\text{cm}} + 2\,\text{cm} = \frac{64\,(\text{cm})^2}{100\,\text{cm}} + 2\,\text{cm} = 2.64\,\text{cm}$$

which we find

$$\sigma_b = 1.3\,\text{MPa}$$

c) The bone will break at the point of the maximum stress, which is at the point where the cross section area of the compact bone layer is minimal. This is the case in the middle of the bone ($x = 0$). At this cross section we have $r_{\text{out}} = 2\,\text{cm}$ and we find in analogy to the above

$$\sigma_{\text{max}} = 1.8\,\text{MPa}$$

Exercise 4

We write the year 1723 and a young soldier who lost his right leg in battle a few months ago is now sitting in front of you. You are the physician who is charged with designing a wooden leg in order to help him walk around again. Or hobble around, that is. As you don't have much experience with prosthetics, you will just have to try what you think is best.

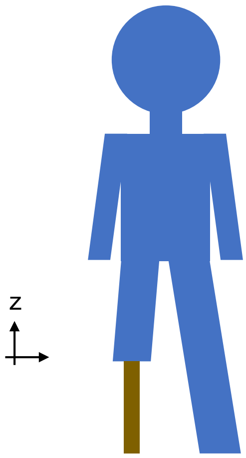

Figure 12: Illustration of the unfortunate soldier with his wooden leg.

a) Your first design is to attach a simple wooden stick to his leg (see figure 12). He weighs $m_2 = 80\,\mathrm{kg}$ but used to weigh $m_1 = 88\,\mathrm{kg}$ before his injury. He is still $s = 1.75\,\mathrm{m}$ tall. The diameter of the stick is $d = 2\,\mathrm{cm}$. Calculate the stress σ_z in the contact area between the leg and the stick.

b) A few hours later, your patient comes back because he feels pain in the area where the prosthesis is attached to his knee. You realize then that the skin can only bear a compressive stress of $\sigma_{skin} = 100\,\mathrm{kPa}$ without pain. What is the diameter of the minimum contact area between the leg and the wooden prosthesis that you should use?

c) The wood you chose is of superior quality and can withstand a maximum compressive stress of $\sigma_{wood} = 1.5\,\mathrm{MPa}$. Knowing this, draw a wooden leg that will be both comfortable and light-weight.

Exercise solution 4

Given: Geometry: $d = 2\,\mathrm{cm}$, mass of the soldier before and after his injury: $m_1 = 88\,\mathrm{kg}$, $m_2 = 80\,\mathrm{kg}$, $\sigma_{skin} = 100\,\mathrm{kPa}$, $\sigma_{wood} = 1.5\,\mathrm{MPa}$.

Asked: Stress exerted by the wooden leg on the patient's knee; minimum contact area; comfortable and light-weight design of a wooden leg.

Relevant relationships:

Definition of stress

$$\sigma = \frac{P}{A}$$

Area of a circle

$$A = \pi r^2$$

a) To calculate the stress between leg and knee, we need to know the cross-sectional area of the leg as well as the force applied on it by the knee. The cross section A is given by

$$A = \pi r^2 = \pi \frac{d^2}{4} = 3.14\,\mathrm{cm}^2$$

Since above the knee, the patient is symmetrical, we can assume that his weight is distributed equally on both knees. The force on one knee is thus given by

$$F_{knee} = \frac{m_{eq.} \cdot g}{2} = \frac{72\,\mathrm{kg} \cdot 9.81\,\mathrm{m\,s^{-2}}}{2} = 353.2\,\mathrm{N}$$

where $m_{eq.}$ is the mass of the man that is supported by the two legs (one real, one wooden). It is obtained by subtracting the mass of the good leg from the total mass of the soldier: $m_{eq.} = m_2 - m_{leg} = m_2 - (m_1 - m_2) = 72\,\mathrm{kg}$.

Finally we can calculate the stress

$$\sigma_z = \frac{F_{knee}}{A} = 1125\,\mathrm{kPa}$$

b) The obtained stress is about 10 times larger than the bearable value. The minimum contact area between the leg and the wooden leg can be found

$$A_{min,knee} = \frac{P}{\sigma_{skin}} = \frac{353.2\,\mathrm{N}}{100\,\mathrm{kPa}} = 35.3\,\mathrm{cm}^2$$

This corresponds to a diameter of

$$d_{min,knee} = \sqrt{A_{min,knee}\frac{4}{\pi}} = 6.7\,\mathrm{cm}$$

c) Since we know the maximum stress that can be put on the wood, we can calculate the minimum diameter of the prosthesis:

$$A_{min,wood} = \frac{P}{\sigma_{wood}} = \frac{353.2\,\mathrm{N}}{1.5\,\mathrm{MPa}} = 2.35\,\mathrm{cm}^2$$

This corresponds to a diameter of

$$d_{min,wood} = \sqrt{A_{min,wood}\frac{4}{\pi}} = 1.73\,\mathrm{cm}$$

With this, we can design a simple wooden leg (see figure 13).

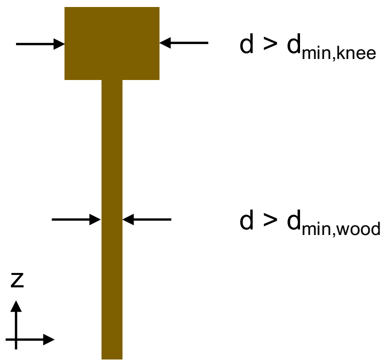

Figure 13: Illustration of a simple wooden leg, with minimum dimensions.

Exercise 5

A force P is acting at the free end of a micro bar and we are measuring the resulting elongation of the bar caused by this force using a resistive strain gauge sensor in a Wheatstone bridge configuration, (see figure 14). The Wheatstone bridge consists of four resistors $R_1 = R_2 = R_3 = R_4 = 600\,\Omega$. The strain gauge R_1 is placed on the micro bar and its resistance varies with strain. The micro bar is made out of silicon and has an initial length of $L = 20\,\mu\text{m}$ (with no force applied). The strain gauge is made of doped silicon and has a gauge factor of $\text{GF} = 30$.

A constant voltage,$V_{cc} = 4.000\,\text{V}$, is applied to the bridge. The voltage measured on the output of the Wheatstone bridge before and after applying the force is $V_{out} = 0.000\,\text{V}$ and $V_{out} = 0.005\,\text{V}$ respectively.

Calculate the length of the cantilever when the force is acting.

hint:

$$V_{out} = \left(\frac{R_3}{R_3 + R_4} - \frac{R_2}{R1 + R_2} \right) \cdot V_{cc}$$

The weight of the microbar is very small compared to the force P and the strain gauge is very thin compared to the microbar.

Exercise solution 5

Given: Change in Voltage, length of beam $L = 20\,\mu\text{m}$, gauge factor $G = 30$, bridge voltage $V_{cc} = 4V$, Values of the resistances of the bridge at rest $R = 600\Omega$

Asked: Length of the cantilever when the force is acting.

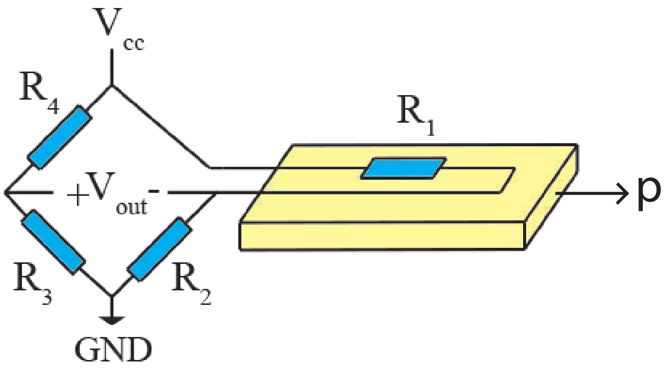

Figure 14: Microbar with strain gauge sensor to measure elongation.

Relevant relationships:

Wheatstone Bridge formula (given)

$$V_{out} = \left(\frac{R_3}{R_3 + R_4} - \frac{R_2}{R1 + R_2} \right) \cdot V_{cc}$$

Definition of gauge factor

$$\mathrm{GF} = \frac{\Delta R}{R} \left(\frac{\Delta L}{L} \right)^{-1}$$

Using the formula of the Wheatstone Bridge with a single active leg, R_1, we note the value of R_1 after deformation R_1' where ($R_1' = R_1 + \Delta R_1$)
The resistance of the remaining 3 legs remains unchanged ($R_2 = R_3 = R_4 = R = 600\Omega$)

$$V_{out} = \left(\frac{R}{2R} - \frac{R}{R_1' + R} \right) \cdot V_{cc}$$

finding R_1'

$$R_1' = R \cdot \frac{V_{cc} + 2V_{out}}{V_{cc} - 2V_{out}}$$

and so

$$\frac{\Delta R_1}{R_1} = \frac{V_{cc} + 2V_{out}}{V_{cc} - 2V_{out}} - 1$$

from the gauge factor equation we know that

$$\Delta L = \frac{L \cdot \frac{\Delta R_1}{R_1}}{GF}$$

Where

$$\frac{\Delta R_1}{R_1}$$

is now known.

The new length after the force is applied is given by

$$L' = L + \Delta L \; = L + \frac{L \cdot \frac{\Delta R_1}{R_1}}{GF}$$

After the numerical application, we find that $L' = 20.003\,\mu\text{m}$