Exercise 1

a) For the vectors

$$\vec{x} = \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 4 \\ c \\ -2 \end{bmatrix}$$

- Calculate the cross product and the dot product of the two vectors.
- Find c such that the vectors $\vec{x}$ and $\vec{y}$ are normal to each other.

b) Give examples for tensors of order zero, one and two.

c) Find the eigenvalues of the tensor

$$\mathbf{\Gamma} = \begin{bmatrix} 10 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Exercise solution 1

Given: Vectors $\vec{x}$ and $\vec{y}$.

Asked: Scalar and cross products, constant c so the vectors are normal to each other.

Relevant relationships:

Dot product

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

where $\vec{a} \cdot \vec{b} = 0$ if $\vec{a}$ and $\vec{b}$ are normal.

Cross product

$$\vec{a} \times \vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Characteristic polynmial of a matrix $\boldsymbol{M}$

$$\det(\boldsymbol{M} - \lambda \mathbb{I}) = 0$$

a) The scalar and cross products are given as

$$\vec{x} \cdot \vec{y} = 6 \cdot 4 - 1 \cdot c + 3 \cdot (-2) = 18 - c$$

and

$$\vec{x} \times \vec{y} = \begin{bmatrix} (-1)\cdot(-2) - 3\cdot c \\ 3\cdot 4 - 6\cdot(-2) \\ 6\cdot c - (-1)\cdot 4 \end{bmatrix} = \begin{bmatrix} 2-3c \\ 24 \\ 6c+4 \end{bmatrix}$$

Since we know that $\vec{x}$ and $\vec{y}$ are normal if their scalar product vanishes, we can solve

$$\vec{x}\cdot\vec{y} = 18 - c = 0 \quad \rightarrow \quad c = 18$$

b) Examples for tensors of order zero (scalar), one (vector) and two (matrix) are

$$\boldsymbol{T}^0 = 1 \qquad \boldsymbol{T}^1 = \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix} \qquad \boldsymbol{T}^2 = \begin{bmatrix} 6 & 2 \\ -4 & 5 \\ 7 & -8 \end{bmatrix}$$

c) Eigenvalues of a matrix are calculated by solving the characteristic polynomial

$$\det(\boldsymbol{\Gamma} - \lambda\mathbb{I}) = \det\left(\begin{bmatrix} 10 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) = \begin{vmatrix} 10-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{vmatrix} = 0$$

The third eigenvalue $\lambda_3 = 0$ follows directly and we only need to look at the rest of the polynomial

$$0 = \begin{vmatrix} 10-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = (10-\lambda)\cdot(2-\lambda) - 3\cdot 3$$
$$= \lambda^2 - 12\lambda + 11$$

for which we find via the quadratic formula

$$\lambda_{1,2} = \frac{12 \pm \sqrt{(-12)^2 - 4\cdot 1\cdot 11}}{2\cdot 1} = \{1, 11\}$$

Exercise 2

When you draw a free body diagram, you sometimes don't know a priori the direction of the forces. You can therefore choose any orientation.

a) Determine the force F_s applied by the spring to the seesaw described figure 1. Use signs consistent with the free body diagram.

b) Determine the force F_s applied by the spring to the seesaw described figure 2. Use signs consistent with the free body diagram.

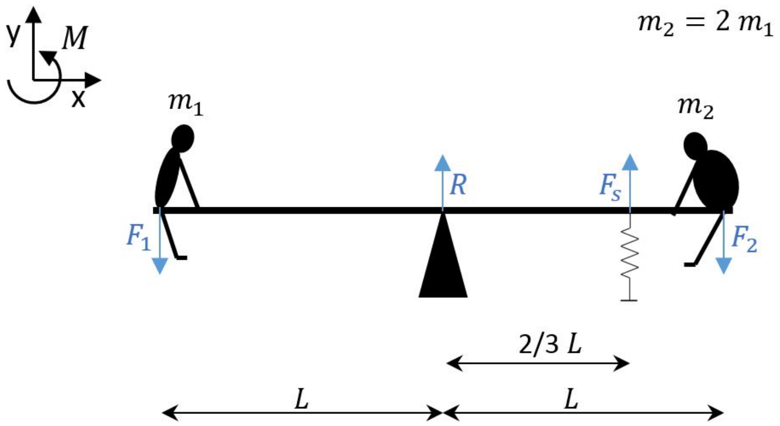

Figure 1: Free body diagram of a seesaw

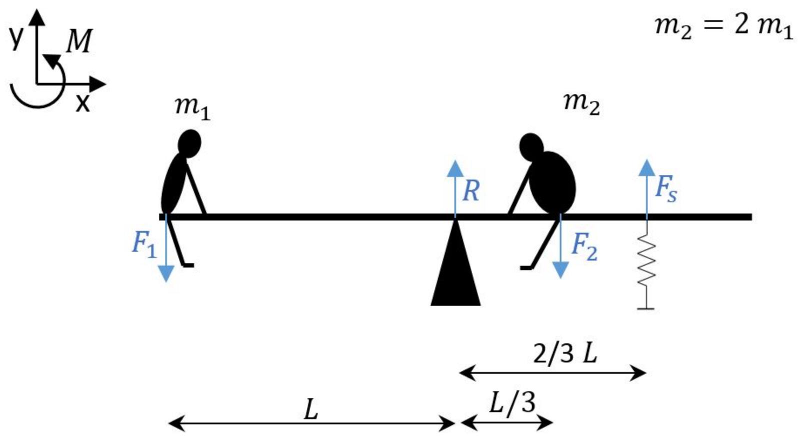

Figure 2: Free body diagram of a seesaw

Exercise solution 2

Given:

distances between the different elements

mass m_1 and $m_2 = 2 \cdot m_1$

Asked: force applied by the spring to the seesaw F_s

Relevant relationships: $\sum \vec{M} = \vec{0}$

a) $\sum M = 0 = m_1 g L - m_2 g L + F_s \cdot \frac{2}{3} L$ so $(1-2) m_1 g + F_s \cdot \frac{2}{3} = 0$. Finally, the force applied by the spring to the seesaw is $F_s = \frac{3}{2} m_1 g$. Since $F_s > 0$, the spring is pushing the seesaw upwards.

b) $\sum M = 0 = m_1 g L - m_2 g L/3 + F_s \cdot \frac{2}{3} L$ so $(1 - 2/3) m_1 g + F_s \cdot \frac{2}{3} = 0$. The force applied by the spring to the seesaw is $F_s = -\frac{1}{2} m_1 g$. Since $F_s < 0$, the spring is pulling the seesaw downwards.

Exercise 3

You are holding a book in your hand; your forearm is horizontal and makes a 90° angle with our upper arm. The weight of the book is 4 kg and the weight of your forearm is 2.5 kg (see figure 3).

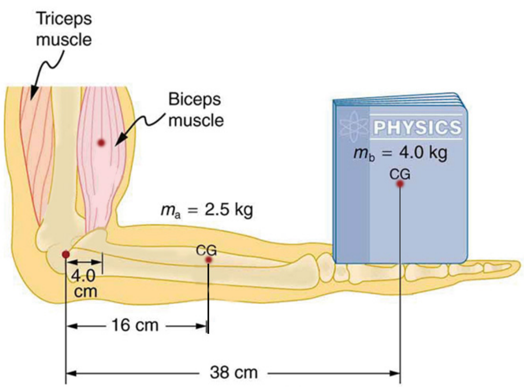

Figure 3: Schematic of the biceps holding up a book.

a) Draw the free body diagram.

b) Calculate the force that is acting on your biceps muscle and the elbow using the distances given in figure 3.

Exercise solution 3

Given:

Mass of the book $m_b = 4.0\,\mathrm{kg}$

Mass of the forearm $m_a = 2.5\,\mathrm{kg}$

Distance between the ellbow and the biceps $r_1 = 4\,\mathrm{cm}$

Distance between the ellbow the center of gravity of the forearm $r_2 = 16\,\mathrm{cm}$

Distance between the ellbow and the book $r_3 = 38\,\mathrm{cm}$

Asked: Free body diagram and forces acting on biceps and elbow.

Relevant relationships:

Sum of forces

$$\sum_i F_i = 0 \qquad i \in x, y, z$$

Sum of moments

$$\sum_i M_i = 0 \qquad i \in x, y, z$$

a) The book held by the hand with the free body diagram is shown in figure **??**. Note that the weight of the book is acting on the center of gravity (CG) of the book. This weight must be multiplied with the gravitational acceleration to obtain the acting force. The same applies for the weight of the forearm.

b) We have four forces that are acting only in the vertical direction. The weight of the book creates a force $w_b = m_b \cdot g$ acting downwards and acts on its center of gravity. The same applies for the forearm. The sum of all forces gives

$$\begin{aligned} &\sum F_y = -F_E + F_B - g \cdot (m_a + m_b) = 0 \\ \rightarrow \quad & F_E = F_B - w_a - w_b \end{aligned} \tag{1}$$

which is underdefined, as both F_E and F_B are unknown. We get an additional equation from the sum of moments at the elbow

$$\begin{aligned} \sum M_E &= -F_E \cdot 0 + F_B \cdot r_1 - r_2 w_a - r_3 w_b = 0 \\ F_B &= \frac{1}{r_1} \cdot (r_2 w_a + r_3 w_b) \\ &= \frac{1}{0.04\,\mathrm{m}} \cdot 9.8\,\mathrm{m\,s^{-2}} \cdot (0.16\,\mathrm{m} \cdot 2.5\,\mathrm{kg} + 0.38\,\mathrm{m} \cdot 4\,\mathrm{kg}) \approx 470\,\mathrm{N} \end{aligned}$$

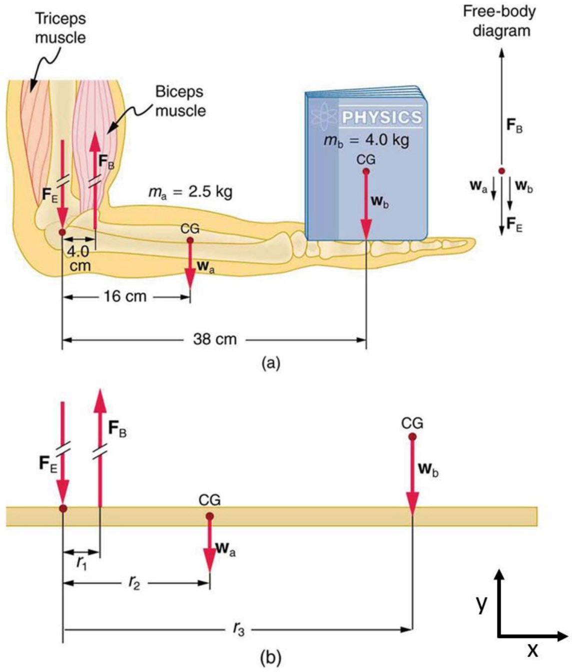

Figure 4: Free body diagram of arm holding a book.

and substituting this into the result from the sum of forces, we get

$$
\begin{aligned}
F_E &= F_B - w_a - w_b \\
&= 470\,\mathrm{N} - 9.8\,\mathrm{m\,s^{-2}} \cdot (2.5\,\mathrm{kg} + 4\,\mathrm{kg}) \approx 410\,\mathrm{N}
\end{aligned}
$$

where F_E and F_B act on the forearm in the directions they were originally introduced as (F_B acting upwards and F_E acting downward).