Exercise 1

A simple beam with a square cross-section (side a, length L) is put under a complex load (fig. 1). The beam is characterised by the following values: $E = 16\,\mathrm{GPa}$, $a = 2\,\mathrm{cm}$, $L = 30\,\mathrm{cm}$ and $F = 100\,\mathrm{N}$.

We define $z = 0$ at the top of the beam and $x = 0$ at the clamped end. Using the principle of superposition and the tables of the formulary, calculate the deflection $\omega(x)$. What is the deflection in $x = L$?

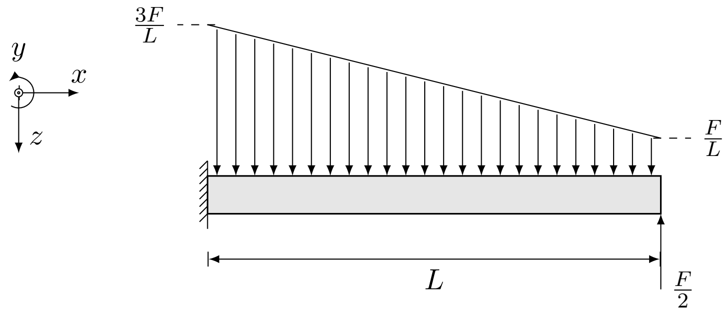

Figure 1: Schematic of the loaded bar and its cross-section.

Exercise solution 1

Given:

- Dimension of the bar: $E = 16\,\mathrm{GPa}$, $a = 2\,\mathrm{cm}$, $L = 30\,\mathrm{cm}$
- Force: $F = 100\,\mathrm{N}$

Asked:

- Deflection $\omega(x)$, numerical value at $x = L$

We have a complex load that can be simplified to a sum of simple loads, for instance a constant distributed load, a linearly decreasing distributed load and a point.

The contribution from the constant distributed load is:

$$\omega_1(x) = \frac{q_1 L^4}{24EI}\left[6\left(\frac{x}{L}\right)^2 - 4\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)^4\right]$$

The contribution from the linearly decreasing distributed load is:

$$\omega_2(x) = \frac{q_2 L^4}{120EI}\left[10\left(\frac{x}{L}\right)^2 - 10\left(\frac{x}{L}\right)^3 + 5\left(\frac{x}{L}\right)^4 + \left(\frac{x}{L}\right)^5\right]$$

The contribution from the point force is:

$$\omega_3(x) = \frac{P_3 L^3}{6EI}\Big[3\Big(\frac{x}{L}\Big)^2 - \Big(\frac{x}{L}\Big)^3\Big]$$

where $q_1 = F/L$, $q_2 = 2F/L$, $P = -F/2$ and $I = a^4/12$.

The total deflection is the sum of all contributions:

$$\begin{aligned}\omega(x) &= \omega_1(x) + \omega_2(x) + \omega_3(x)\\ &= \frac{FL^3}{10Ea^4}\Big[20\Big(\frac{x}{L}\Big)^2 - 30\Big(\frac{x}{L}\Big)^3 + 15\Big(\frac{x}{L}\Big)^4 - 2\Big(\frac{x}{L}\Big)^5\Big]\end{aligned}$$

At the free end of the beam, the deflection is thus:

$$\omega(L) = 1.05 \cdot 10^{-4} \cdot (20 - 30 + 15 + 2) = 0.32mm$$

Exercise 2

A steel bar having a square cross section (50 mm × 50 mm) and length $L = 2\,\mathrm{m}$ is compressed by axial loads that have a resultant $P = 60\,\mathrm{kN}$ acting at the midpoint of one side of the cross section (see fig. 2).
Assuming that the modulus of elasticity E is equal to 210 GPa and that the ends of the bar are pinned, calculate the maximum deflection δ and the maximum bending moment M_{max}.

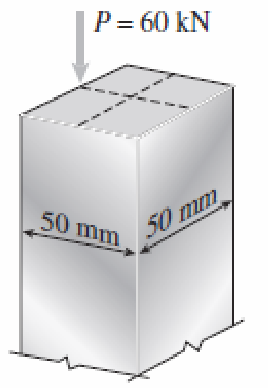

Figure 2: Square cross section bar and load P

Exercise solution 2

Given:

- Dimension of the bar: 50 mm × 50 mm × 2 m
- Axial load $P = 60\,\mathrm{kN}$
- Young's modulus of the bar $E = 210\,\mathrm{GPa}$

Asked:

- Maximum deflection δ
- Maximum bending moment $M_{\max}$

Relevant relationships:

- For a bar with the square cross section and side equals to b : $I = \frac{b^4}{12}$
- Maximum deflection $\delta{=}e\left[\sec\left(\frac{L}{2}\sqrt{\left(\frac{P}{EI}\right)}\right) - 1\right]$
- M=$p \times e$

Second moment of area for the square cross section:

$$I = \frac{b^4}{12} = 520.8 \times 10^3\,\mathrm{mm}^4 \tag{1}$$

so maximum deflection is given by:

$$\delta = e\left[\sec\left(\frac{L}{2}\sqrt{\left(\frac{P}{EI}\right)}\right) - 1\right] = 8.87\,\mathrm{mm} \tag{2}$$

and the maximum bending moment is

$$M_{\max} = Pe\left[\sec\left(\frac{L}{2}\sqrt{\left(\frac{P}{EI}\right)}\right) - 1 + 1\right] = Pe\left[\sec\left(\frac{L}{2}\sqrt{\left(\frac{P}{EI}\right)}\right)\right] = 2.03\,\mathrm{kN\,m} \tag{3}$$

+1 in the bending moment formula is because the load acts on $e + \delta$ so e should be added to the total deflection.

Exercise 3

The truss ABC (fig. 3) supports a vertical load W at joint B. Each member is a slender circular steel pipe ($E = 200\,\mathrm{GPa}$) with outside diameter 100 mm and wall thickness 6 mm. The distance between supports is 7 m. Joint B is restrained against displacement perpendicular to the plane of the truss. Determine the critical value W_{cr} of the load.

Hint: Second moment of area for the pipe of outer and inner diameter d_o and d_i respectively, is $I = \frac{\pi}{64}\left(d_o^4 - d_i^4\right)$.

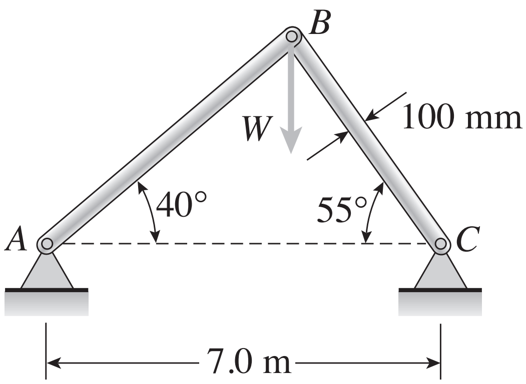

Figure 3: Truss structure under load

Exercise solution 3

Given:

- Pipes' material Young's modulus $E = 200\,\text{GPa}$
- Distance between pin supports $L = 7\,\text{m}$
- Outer pipe diameter $d_o = 100\,\text{mm}$
- Pipe wall thickness $t = 6\,\text{mm}$

Asked:

- Critical value of the load W_{cr}, for truss not to buckle

Relevant relationships:

- Second moment of area for the pipe $I = \frac{\pi}{64}\left(d_o^4 - d_i^4\right)$
- writing the equilibrium equation for the forces on each axis $\sum F_h = 0$ and $\sum F_v = 0$
- Euler's formula for critical buckling force

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

- effective length for the pinned-pinned end conditions $L_e = L$

Solution
Inner diameter of the pipe is $d_i = d_o - 2t = 88\,\text{mm}$, so second moment of area is

$$I = \frac{\pi}{64}\left(d_o^4 - d_i^4\right) = 1.965 \times 10^6\,\text{mm}^4 \tag{4}$$

Lengths of the pipes AB and BC can be calculated from the geometry. If we look at the figure 4. we can see that $L_1 + L_2 = h\tan(50^o) + h\tan(35^o) = L$, so we calculate the height h and then L_{AB} and L_{BC} as

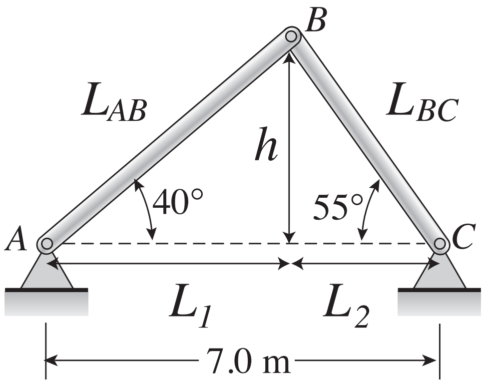

Figure 4: Truss structure under load

$$h = \frac{L}{\tan(50^o) + \tan(35^o)} \tag{5}$$

$$L_{AB} = h/\cos(50^o) = \frac{L}{\tan(50^o) + \tan(35^o)} \cdot \frac{1}{\cos(50^o)} = 5.756\,\mathrm{m} \tag{6}$$

$$L_{BC} = h/\cos(35^o) = \frac{L}{\tan(50^o) + \tan(35^o)} \cdot \frac{1}{\cos(35^o)} = 4.517\,\mathrm{m} \tag{7}$$

Buckling occurs when either member reaches its critical load:

$$P_{AB}^{cr} = \frac{\pi^2 EI}{L_{AB}^2} = 117.1\,\mathrm{kN} \tag{8}$$

$$P_{BC}^{cr} = \frac{\pi^2 EI}{L_{BC}^2} = 190.1\,\mathrm{kN} \tag{9}$$

If we look at the truss sketch, presented on the figure 5. and write equilibrium equations for all horizontal and vertical forces we have

$$\sum F_{horiz} = 0 \Rightarrow F_{AB}\sin(50^o) - F_{BC}\sin(35^o) = 0 \tag{10}$$

$$\sum F_{vert} = 0 \Rightarrow F_{AB}\cos(50^o) + F_{BC}\cos(35^o) - W = 0 \tag{11}$$

Solving the equations we get

$$W = 1.7386 \cdot F_{AB} \tag{12}$$

$$W = 1.3004 \cdot F_{BC} \tag{13}$$

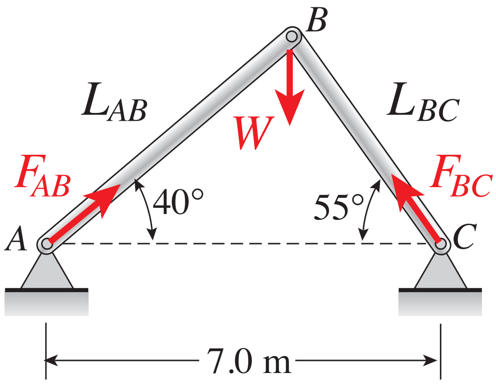

Figure 5: Truss structure under load

Critical value of the load W based on the member AB is

$$W_{AB}^{cr} = 1.7386 \cdot P_{AB}^{cr} = 203\,\text{kN} \tag{14}$$

Critical value of the load W based on the member BC is

$$W_{BC}^{cr} = 1.3004 \cdot P_{BC}^{cr} = 247\,\text{kN} \tag{15}$$

So, the truss will buckle at the $W_{cr} = min(W_{AB}^{cr}, W_{BC}^{cr}) = 203\,\text{kN}$, when member AB buckles.