Exercise 1

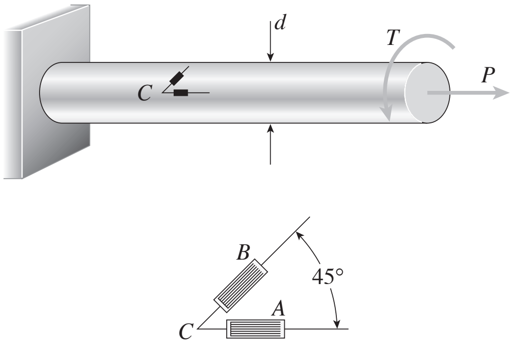

Figure 1: Circular bar and strain gauges A and B

A solid circular bar with a diameter of $d = 32\,\text{mm}$ is subjected to an axial force P and a torque T (see Figure 1). Strain gauges A and B mounted on the surface of the bar give readings $\varepsilon_A = 140 \times 10^{-6}$ and $\varepsilon_B = -60 \times 10^{-6}$. The bar is made of steel having $E = 210\,\text{GPa}$ and $\nu = 0.29$.

a) Determine the axial force P and the torque T.

b) Determine the maximum shear strain γ_{max} and the maximum shear stress τ_{max} in the bar.

Exercise solution 1

Given:

- Strains on the strain gauges $\varepsilon_A = 140 \times 10^{-6}$ and $\varepsilon_B = -60 \times 10^{-6}$.
- Diameter of the bar $d = 32\,\text{mm}$.
- Material properties $E = 210\,\text{GPa}$, $\nu = 0.29$.

Asked:

- Axial force P and torque T.
- Maximum shear stress and maximum shear strain.

Relevant relationships:

- $\sigma = \frac{P}{A}$.
- For a circular shaft : $\tau = \frac{16T}{\pi d^3}$

- Transformation equation :

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos(2\theta) + \frac{\gamma_{xy}}{2}\sin(2\theta)$$

a)

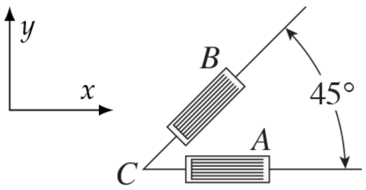

Figure 2: Strain gauges A and B, definition of the axes

For an element at the surface of the cylinder (at point C), we define our axes according to fig. 2.

At $\theta = 0°$: $\varepsilon_A = \varepsilon_x = 140 \times 10^{-6}$
At $\theta = 45°$: $\varepsilon_B = -60 \times 10^{-6}$

The element is in plane stress:

$$\sigma_x = \frac{P}{A} = \frac{4P}{\pi d^2}$$

$$\sigma_y = 0$$

$$\tau_{xy} = -\frac{16T}{\pi d^3}$$

$$\varepsilon_x = 140 \times 10^{-6},\ \varepsilon_y = -\nu\varepsilon_x = -40.6 \times 10^{-6}$$

From this, we can already find the axial force P:

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{4P}{\pi d^2 E}$$

$$P = \frac{\pi d^2 E \varepsilon_x}{4} = 23.632\,\text{kN}$$

We can also express the shear strain as a function of T:

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\nu)\tau_{xy}}{E} = -\frac{32T(1+\nu)}{\pi d^3 E} = -(1.91 \times 10^{-6})T$$

To find the value of the shear strain, we apply a strain transformation for $\theta = 45°$:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos(2\theta) + \frac{\gamma_{xy}}{2}\sin(2\theta)$$

$$\varepsilon_{x_1} = \varepsilon_B = -60 \times 10^{-6}$$

$$2\theta = 90^\circ$$

We obtain:

$$-60 \times 10^{-6} = 49.7 \times 10^{-6} - (0.955 \times 10^{-6})T$$

$$T = 114.86\,\mathrm{N\,m}$$

b)

To find the shear strain, we can directly use the formula:

$$\gamma_{xy} = -(1.91 \times 10^{-6})T = 219.4 \times 10^{-6}$$

$$\frac{\gamma_{max}}{2} = (\varepsilon_{x'y'})_{max} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\gamma_{max} = 2\,84.17 \times 10^{-6}$$

With Hooke's law, we find the maximum shear stress:

$$\tau_{max} = G\gamma_{max} = 23.12\,\mathrm{MPa}$$

Exercise 2

Reminder about non overconstrained systems : There is no need to use the displacement stiffness method when the system is not overconstrained.

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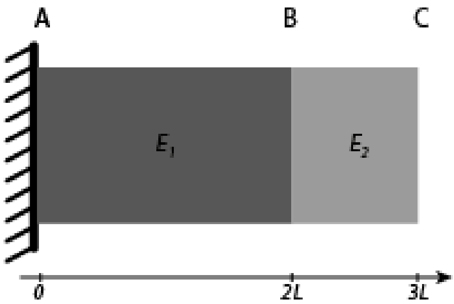
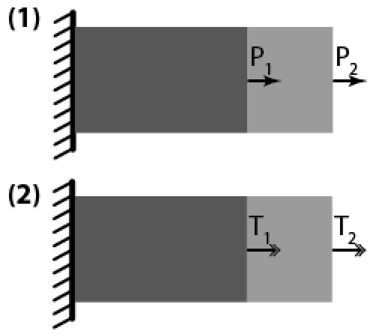
We consider a cylindrical beam made out of two materials as depicted Fig.3a. Its radius is r, young modulus are E_1, E_2 and Poisson ratio are ν_1, ν_2 .



(a) Beam made out of two materials



(b) Two configurations

Figure 3

a) For the configuration 1 in Fig.3a (forces),
   - What is the internal force in the beam ?
   - What is and where is the maximum stress ?
   - What is the displacement of points A, B and C ?

b) For the configuration 2 in Fig.3b (torques),
   - What is the internal torque in the beam ?
   - What is and where is the maximum shear stress ?
   - What is the angle of points A, B and C ?

Exercise solution 2

a)

- **What is the internal force in the beam ?** With the method of sections, we deduce that the internal force is $P_1 + P_2$ between A and B, and P_2 between B and C.

- **What is and where is the maximum stress ?** The stress between A and B is uniform and its value is $\sigma_{1,max} = \sigma_1 = \frac{P_1+P_2}{\pi r^2}$. The stress between B and C is uniform and its value is $\sigma_{2,max} = \sigma_2 = \frac{P_2}{\pi r^2}$. As a consequence, the stress is maximum in the whole part of the beam between points A and B, and its value is $\sigma_{max,total} = \frac{P_1+P_2}{\pi r^2}$.

- **What is the displacement of points A, B and C ?** $u_A = 0$ (clamped beam). $u_B = u_A + \int_0^{2L} \varepsilon(x)dx = \int_0^{2L} \sigma_1/E_1 dx = \frac{2L(P_1+P_2)}{E_1 \pi r^2}$. $u_C = u_B + \int_{2L}^{3L} \sigma_2/E_2 dx = \frac{L \cdot P_2}{E_2 \pi r^2} + \frac{2L(P_1+P_2)}{E_1 \pi r^2}$.

b)

It is exactly the same method, where P becomes T, E becomes $G = \frac{E}{2(1+\nu)}$, $A = \pi r^2$ becomes $J = \frac{\pi}{2}\pi(d/2)^4 = \frac{\pi d^4}{32}$, $\sigma = \frac{P}{A}$ becomes $\tau(r) = \frac{T \cdot r}{J}$, $\sigma = E\varepsilon$ becomes $\tau = G\gamma$ and $u = \frac{PL}{EA}$ becomes $\varphi = \frac{TL}{GJ}$.

- **What is the internal torque in the beam ?** With the method of sections, we deduce that the internal force is $T_1 + T_2$ between A and B, and T_2 between B and C.

- **What is and where is the maximum shear stress ?** The shear stress in a beam under torsion is non uniform : for a cylindrical beam, it is proportional to the radius : $\tau = \frac{T \cdot r}{J}$. The shear stress is therefore maximum at $r = d/2$. Between A and B, the maximum shear value is $\tau_{1,max} = \frac{(T_1+T_2)d/2}{J} = \frac{16(T_1+T_2)}{\pi d^3}$. The stress between B and C is $\tau_{2,max} = \frac{16T_2}{\pi d^3}$. As a consequence, the stress is maximum at $r = d/2$ in the part of the beam between points A and B, and its value is $\tau_{max,total} = \frac{16(T_1+T_2)}{\pi d^3}$.

- **What is the angle of twist at points A, B and C ?** $\varphi_A = 0$ (clamped beam). $\varphi_B = \frac{(T_1+T_2)2L}{G_1 J} = \frac{64L(T_1+T_2)(1+\nu_1)}{E_1 \pi d^4}$ and $\varphi_C = \varphi_B + \frac{T_2 L}{G_2 J} = \frac{64L(T_1+T_2)(1+\nu_1)}{E_1 \pi d^4} + \frac{32LT_2(1+\nu_2)}{E_2 \pi d^4}$.