Exercise 1

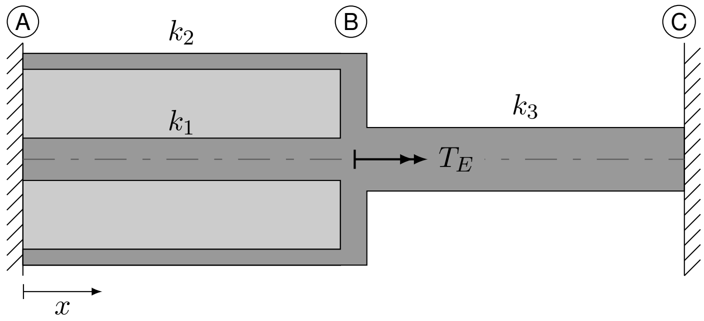


Figure 1: Cross section through a bar and tube system under torsion.

The torsion bar system in figure 2 consists of a hollow tube and two different bars connected in the middle with a stiff plate. The system is being twisted in the middle with an external torque T_E.

a) Find the twist at point B and the reaction torques at the walls A and C with the displacement stiffness method.

b) Calculate the reaction torque of the wall against the tube for $T_E = 1\,\mathrm{N\,m}$, $G = 10\,\mathrm{MPa}$ and lengths $L_{AB} = L_{BC} = 1\,\mathrm{m}$. Assume axially symmetric cross sections with a diameter $d_1 = 3\,\mathrm{cm}$ for the bar inside the tube, an outer tube diameter of $d_{2,o} = 10\,\mathrm{cm}$ and a tube wall thickness of $t_2 = 5\,\mathrm{mm}$, as well as a diameter $d_3 = 5\,\mathrm{cm}$ for the bar on the right.

Exercise solution 1

Given: Geometry, applied torque. Values to enter in part b).

Asked: Twist in the middle of the structure, reaction torques.

Relevant relationships:

Torsion formula

$$T = \frac{GJ \cdot \varphi}{L} = k \cdot \varphi$$

Second moment of area for a circle

$$J_c = \frac{\pi d^4}{32}$$

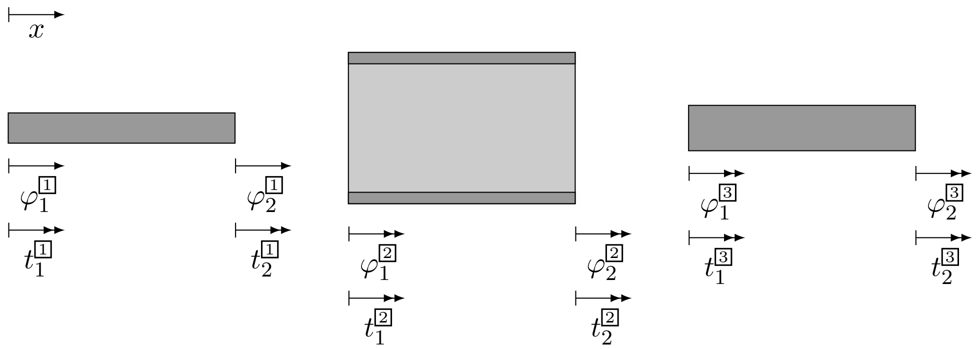


Figure 2: Torsion bar system with isolated parts and local torques and angles.

a)

1. We find the local stiffness matrices as

$$\begin{bmatrix} t_1^{\boxed{1}} \\ t_2^{\boxed{1}} \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{bmatrix} \varphi_1^{\boxed{1}} \\ \varphi_2^{\boxed{1}} \end{bmatrix}$$
$$\begin{bmatrix} t_1^{\boxed{2}} \\ t_2^{\boxed{2}} \end{bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} \varphi_1^{\boxed{2}} \\ \varphi_2^{\boxed{2}} \end{bmatrix}$$
$$\begin{bmatrix} t_1^{\boxed{3}} \\ t_2^{\boxed{3}} \end{bmatrix} = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{bmatrix} \varphi_1^{\boxed{3}} \\ \varphi_2^{\boxed{3}} \end{bmatrix}$$

The mapping of local to global variables is

$$\begin{aligned} \varphi_1 &= \varphi_1^{\boxed{1}} & T_1 &= t_1^{\boxed{1}} \\ \varphi_2 &= \varphi_1^{\boxed{2}} & T_2 &= t_1^{\boxed{2}} \\ \varphi_3 &= \varphi_2^{\boxed{1}} = \varphi_2^{\boxed{2}} = \varphi_1^{\boxed{3}} & T_3 &= t_2^{\boxed{1}} + t_2^{\boxed{2}} + t_1^{\boxed{3}} \\ \varphi_4 &= \varphi_2^{\boxed{3}} & T_4 &= t_2^{\boxed{3}} \end{aligned}$$

2. We derive the global stiffness matrix by either writing out the matrix directly line–by–line as

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} t_1^{\boxed{1}} \\ t_1^{\boxed{2}} \\ t_2^{\boxed{1}} + t_2^{\boxed{2}} + t_1^{\boxed{3}} \\ t_2^{\boxed{3}} \end{bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_2 & -k_2 & 0 \\ -k_1 & -k_2 & k_1+k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix}$$

or by first expanding the local stiffness matrices by finding the proper lines for the $t_i^{\boxed{j}}$ and filling the rest with zeroes. In the same manner we fill the columns for the φ_k that do not appear in the local matrix with zeros.

$$\begin{bmatrix} t_1^{\boxed{1}} \\ t_2^{\boxed{1}} \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_3 \end{bmatrix} \implies \begin{bmatrix} t_1^{\boxed{1}} \\ 0 \\ t_2^{\boxed{1}} \\ 0 \end{bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix}$$

$$\begin{bmatrix} t_1^{\boxed{2}} \\ t_2^{\boxed{2}} \end{bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} \varphi_2 \\ \varphi_3 \end{bmatrix} \implies \begin{bmatrix} 0 \\ t_1^{\boxed{2}} \\ t_2^{\boxed{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix}$$

$$\begin{bmatrix} t_1^{\boxed{3}} \\ t_2^{\boxed{3}} \end{bmatrix} = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{bmatrix} \varphi_3 \\ \varphi_4 \end{bmatrix} \implies \begin{bmatrix} 0 \\ 0 \\ t_1^{\boxed{3}} \\ t_2^{\boxed{3}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix}$$

3. The global stiffness equation can be rewritten using the boundary conditions (all twists except the one in the middle are zero) as

$$\begin{bmatrix} T_1 \\ T_2 \\ T_E \\ T_4 \end{bmatrix} = \begin{bmatrix} t_1^{\boxed{1}} \\ t_1^{\boxed{2}} \\ t_2^{\boxed{1}} + t_2^{\boxed{2}} + t_1^{\boxed{3}} \\ t_2^{\boxed{3}} \end{bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_2 & -k_2 & 0 \\ -k_1 & -k_2 & k_1+k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \varphi_3 \\ 0 \end{bmatrix}$$

4. We solve for φ_3 using the third line to get the unknown twist

$$\varphi_3 = \frac{T_E}{k_1 + k_2 + k_3}$$

5. Finally we find the reaction torques from the remaining lines of the matrix as

$$T_1 = -\frac{k_1 \cdot T_E}{k_1 + k_2 + k_3} \qquad T_2 = -\frac{k_2 \cdot T_E}{k_1 + k_2 + k_3} \qquad T_4 = -\frac{k_3 \cdot T_E}{k_1 + k_2 + k_3}$$

b)

Using the previous result, we put in the second moment of area formulas

$$k_1 = \frac{G}{L} \cdot \frac{\pi d_1^4}{32} \qquad k_2 = \frac{G}{L} \cdot \frac{\pi (d_{2,o}^4 - (d_{2,o} - 2t_2)^4)}{32} \qquad k_3 = \frac{G}{L} \cdot \frac{\pi d_3^4}{32}$$

and find that, quite conveniently, the factor $\frac{G\pi}{32L}$ falls away and we get

$$T_1 = -\frac{d_1^4}{d_1^4 + d_{2,o}^4 - (d_{2,o} - 2t_2)^4 + d_3^4} \cdot T_E \approx -0.02\,\mathrm{N\,m}$$

$$T_2 = -\frac{d_{2,o}^4 - (d_{2,o} - 2t_2)^4}{d_1^4 + d_{2,o}^4 - (d_{2,o} - 2t_2)^4 + d_3^4} \cdot T_E \approx -0.83\,\mathrm{N\,m}$$

$$T_4 = -\frac{d_3^4}{d_1^4 + d_{2,o}^4 - (d_{2,o} - 2t_2)^4 + d_3^4} \cdot T_E \approx -0.15\,\mathrm{N\,m}$$

We check our calculation by verifying that $\sum T = T_1 + T_2 + T_E + T_4 = 0$ as we are at equilibrium.

Exercise 2

Two torques are applied to the structure defined figure 3. We want to determine the torques applied to the structure by the wall as well as the angles of twist in the structure. The structure is made of a homogeneous material with a modulus of rigidity G.

a) Define the nodes and elements. Reminder : each node corresponds to a unique torque and a unique angle.

b) Write the local stiffness matrices as a function of $k = \frac{\pi R^4 G}{L}$.

c) Assemble the local stiffness matrices into the global stiffness equation in which you will also incorporate the boundary conditions.

d) Solve the global stiffness equation and express the unknown variables as a function of k, T_A, T_B.

e) Plot the angle of twist of the tube and the cylinder along the x axis if $T_A = T_B = T$.

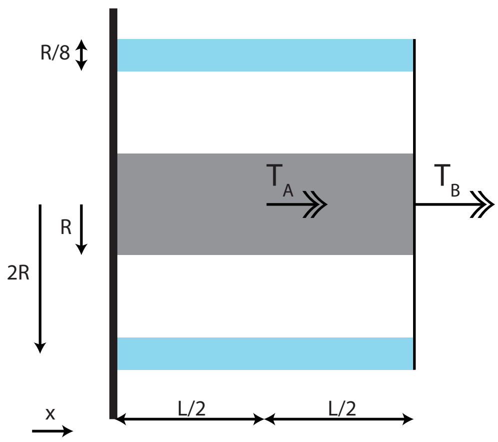


Figure 3: Cross section through a bar and tube system under torsion.

Exercise solution 2

Given: Geometry, applied torques.

Asked: Twist in the structure, reaction torques.

Relevant relationships:

Torsion formula $T = \frac{GJ \cdot \varphi}{L} = k \cdot \varphi$

Second moment of area for a circle $J_c = \frac{\pi r^4}{2}$

Second moment of area for a thin tube $J_t = 2\pi r^3 t$

a)

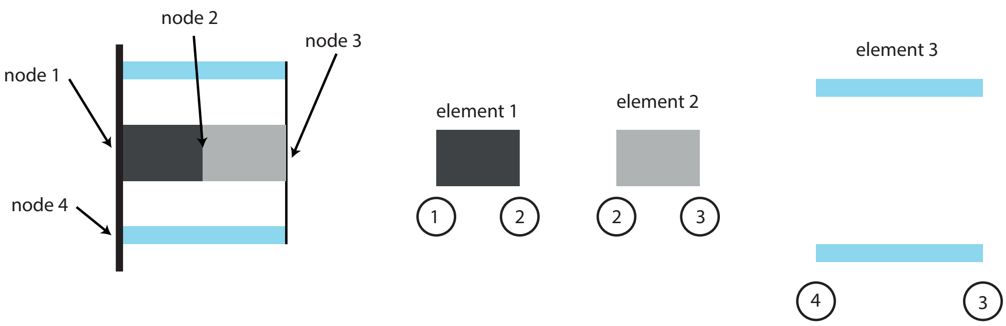


Figure 4: Definition of the nodes and the elements

We need to define four nodes for this system. Node 1 and node 4 cannot be merged as the torque applied by the wall to node 1 is different from the one applied to node 4.

b)

The torsionnal stiffnesses are :

- $k_1 = \frac{GJ_c}{L/2}$ with $J_c = \frac{\pi R^4}{2}$ so $k_1 = \frac{G\pi R^4}{L} = k$
- $k_2 = k$ (same method)
- $k_3 = \frac{GJ_t}{L/2}$ with $J_t = 2\pi(2R)^3 R/8 = \pi R^4$ so $k_3 = 2\frac{G\pi R^4}{L} = 2k$

We find the local stiffness matrices as

$$\begin{bmatrix} t_1^{\boxed{1}} \\ t_2^{\boxed{1}} \end{bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \varphi_1^{\boxed{1}} \\ \varphi_2^{\boxed{1}} \end{bmatrix}$$
$$\begin{bmatrix} t_2^{\boxed{2}} \\ t_3^{\boxed{2}} \end{bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \varphi_2^{\boxed{2}} \\ \varphi_3^{\boxed{2}} \end{bmatrix}$$
$$\begin{bmatrix} t_4^{\boxed{3}} \\ t_3^{\boxed{3}} \end{bmatrix} = k \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \varphi_4^{\boxed{3}} \\ \varphi_3^{\boxed{3}} \end{bmatrix}$$

c)

The mapping of local to global variables is

$$\varphi_1 = \varphi_1^{\boxed{1}} \qquad T_1 = t_1^{\boxed{1}}$$
$$\varphi_2 = \varphi_2^{\boxed{1}} = \varphi_2^{\boxed{2}} \qquad T_2 = t_2^{\boxed{1}} + t_2^{\boxed{2}}$$
$$\varphi_3 = \varphi_3^{\boxed{2}} = \varphi_3^{\boxed{3}} \qquad T_3 = t_3^{\boxed{2}} + t_3^{\boxed{3}}$$
$$\varphi_4 = \varphi_4^{\boxed{3}} \qquad T_4 = t_4^{\boxed{3}}$$

We derive the global stiffness matrix by either writing out the matrix directly line–by–line

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} t_1^{\boxed{1}} \\ t_2^{\boxed{1}} + t_2^{\boxed{2}} \\ t_3^{\boxed{2}} + t_3^{\boxed{3}} \\ t_4^{\boxed{3}} \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix}$$

The global stiffness equation can be rewritten using the boundary conditions $\varphi_1 = 0$, $\varphi_4 = 0$, $T_2 = T_A$, $T_3 = T_B$ as

$$\begin{bmatrix} T_1 \\ T_A \\ T_B \\ T_4 \end{bmatrix} = k \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ \varphi_2 \\ \varphi_3 \\ 0 \end{bmatrix}$$

d)

We solve for the angles φ_2 and φ_3 using the second and third line to get the unknown twist

$$\begin{aligned} T_A &= k(2\varphi_2 - \varphi_3) \\ 3T_A + T_B &= 5k\varphi_2 \\ \varphi_2 &= \frac{3}{5k}T_A + \frac{1}{5k}T_B \end{aligned} \qquad \begin{aligned} T_B &= k(-\varphi_2 + 3\varphi_3) \\ T_A + 2T_B &= 5k\varphi_3 \\ \varphi_3 &= \frac{1}{5k}T_A + \frac{2}{5k}T_B \end{aligned}$$

Finally we find the reaction torques from the remaining lines of the matrix as

$$T_1 = -k\varphi_2 = -\frac{3}{5}T_A - \frac{1}{5}T_B \qquad T_4 = -2k\varphi_3 = -\frac{2}{5}T_A - \frac{4}{5}T_B$$

We check our calculation by verifying that $\sum T = T_1 + T_A + T_B + T_4 = 0$ as we are at equilibrium.

e)

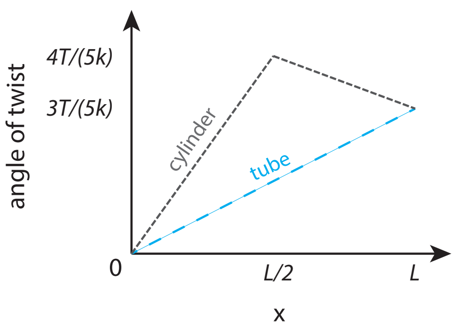


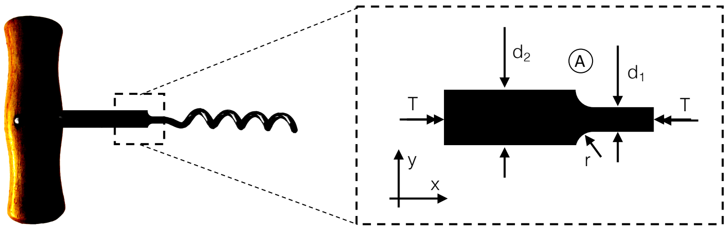
Figure 5: Angle of twist as a function of x in the cylinder and in the tube

Exercise 3

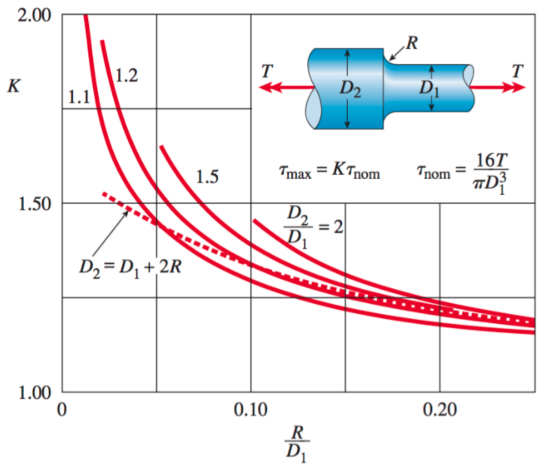
You have just learned that you got a great mark on your Structural Mechanics exam. Your friend Santiago however has a red face and upon asking, you find out that he failed miserably. Since you both have ample reason to drink, you decide to open a bottle of wine.

a) In his frustration, Santiago puts too much torque on his vintage corkscrew and it breaks at point A (fig. 6a). Calculate the maximum torque that he should not have exceeded. Use $d_1 = 1\text{mm}$, $d_2 = 1.5\text{mm}$, $r = 0.05\text{mm}$. The ultimate shear strength of the material is $\tau_U = 200\text{N/mm}^2$.

b) You take the bottle and use your patented Swiss army knife. Without surprise, you have no trouble opening the bottle. Knowing that driving a corkscrew into a cork results in a maximal torque of 0.1 Nm, calculate the safety factor that the manufacturer used on the army knife. The dimensions are $d_1 = 1.5\text{mm}$, $d_2 = 3\text{mm}$, $r = 0.35\text{mm}$. The material is steel, with $\tau_U = 360\text{N/mm}^2$.



(a) Illustration of the corkscrew: problem definition.



(b) Torsional stress concentration factor as a function of the geometry.

Figure 6: Illustration of the corkscrew and stress concentration factor graph.

Exercise solution 3

Given: Geometries, ultimate strength, maximum torque on the cork.

Asked: Maximum torque for the vintage corkscrew, safety factor on the Swiss army knife.

Relevant relationships:

Maximum shear stress (stress concentration)

$$\tau_{max} = K \cdot \frac{Tc}{J}$$

Second moment of interia for a circle

$$J_c = \frac{\pi c^4}{2}$$

a)

First, we need to find K using the stress concentration table (fig.6a). We have $d_2/d_1 = 1.5$ and $r/d_1 = 0.05$. Thus we find that $K \approx 1.7$.
The maximum shear stress that the screw can withstand is thus given by:

$$\tau_{max} = K \cdot \frac{T_{max}c}{J} = 2K \cdot \frac{T_{max}}{\pi c^3}$$

We can now isolate the torque:

$$T_{max} = \frac{\pi c^3 \tau_{max}}{2K} = \frac{\pi \cdot (0.5 \cdot 10^{-3})^3[m^3] \cdot 200 \cdot 10^6[N/m^2]}{2 \cdot 1.7} = 0.023Nm$$

b)

In this case, we are looking for the safety factor that was used by the manufacturer: S = σ_U/σ_{max}. The ultimate shear strength is a given material property. The maximum stress can be calculated similarly to part (a). First, we again find K for our geometry. We have $d_2/d_1 = 2$ and $r/d_1 = 0.23$. Thus we find that $K \approx 1.2$.
The maximum shear stress that can be applied is thus:

$$\tau_{max} = 2K \cdot \frac{T_{max}}{\pi c^3} = 2 \cdot 1.2 \cdot \frac{0.1[Nm]}{\pi(0.75 \cdot 10^{-3})^3[m^3]} = 1.8 \cdot 10^8 N/m^2$$

The safety factor is therefore:

$$S = \frac{\sigma_U}{\sigma_{max}} = \frac{360[N/mm^2]}{180[N/mm^2]} = 2$$

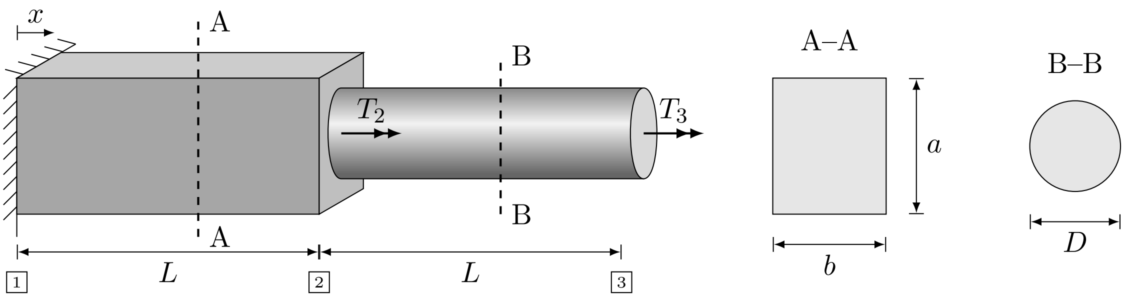


Figure 7: Bar system under torsion and associated cross-sections

Exercise 4

When bars under torsion have a non–circular cross–section, one can use correction factors to find the maximum shear stress $\tau_{\max}$ and the twist φ in those beams. The bar system under torsion shown in figure 7 consists of one bar with rectangular cross-section that is fixed to a wall, as well as a round bar attached to the rectangular bar. Both bars are made out of aluminium (Young's modulus $E = 70\,\text{GPa}$, Poisson ratio $\nu = 0.35$). The bars have a length of $L = 40\,\text{cm}$ each. The other dimensions are $a = 2.4\,\text{cm}$, $b = 2\,\text{cm}$ and $D = 1.8\,\text{cm}$. The torques applied are $T_2 = 1.0\,\text{N m}$ and $T_3 = 2.0\,\text{N m}$.

a) What is the internal torque in the cylinder? And in the rectangular bar?

b) Find the maximum shear stress in the whole system.

c) What is the value of the shear modulus G for aluminium ? Find the angle of twist in point 2 and in point 3.

Exercise solution 4

a) We use the method of sections to find the internal torques in the bars.

The internal torque in the rectangular member is a sum of the two applied torques:

$$T(x) = T_2 + T_3, \quad 0 \leq x \leq L$$

The internal torque in the round member is simply T_3:

$$T(x) = T_3, \quad L \leq x \leq 2L$$

b) Let's now find the maximum shear stress in the structure. We have to calculate the maximum stress for each bar separately and then chose the higher one.

Global maximum shear stress (Rectangular member)

Ratio of side lengths:

$$\frac{a}{b} = \frac{2.4\,\mathrm{cm}}{2\,\mathrm{cm}} = 1.2 \quad \rightarrow \quad C_1 = 0.219, \quad C_2 = 0.1661$$

Maximum shear stress in point B directly from the formulas:

$$\tau_{\max} = \frac{T_2 + T_3}{C_1 \cdot a \cdot b^2} = \frac{3\,\mathrm{N\,m}}{0.219 \cdot 2.4\,\mathrm{cm} \cdot (2\,\mathrm{cm})^2} = 1.43\,\mathrm{MPa}$$

Global maximum shear stress (Round member)

The polar second moment of area is given by:

$$J = \frac{\pi c^4}{2} = \frac{\pi\,(0.9\,\mathrm{cm})^4}{2} = 1.03\,\mathrm{cm}^4$$

The maximum shear stress (formula sheet) is:

$$\tau_{\max} = \frac{T_3 c}{J} = \frac{2\,\mathrm{N\,m} \cdot 0.9\,\mathrm{cm}}{1.03\,\mathrm{cm}^4} = 1.75\,\mathrm{MPa}$$

which is also the global maximum.

c) We are given Young's modulus, but we need shear modulus for torsion:

$$G = \frac{E}{2 \cdot (1+\nu)} = 25.9\,\mathrm{GPa}$$

In the rectangular member we get the twist at point 2:

$$\begin{aligned} \varphi_2 &= \varphi_1 + \frac{(T_2 + T_3) \cdot L}{C_2 \cdot a \cdot b^3 \cdot G} \\ &= 0 + \frac{3\,\mathrm{N\,m} \cdot 40\,\mathrm{cm}}{0.1661 \cdot 2.4\,\mathrm{cm} \cdot (2\,\mathrm{cm})^3 \cdot 25.9\,\mathrm{GPa}} \\ &= 1.45 \times 10^{-3}\,\mathrm{rad} = 0.083° \end{aligned}$$

The twist in point 3 is the sum of the twist in point 2 and the additional twist from the round member:

$$\begin{aligned} \varphi_3 = \varphi_2 + \frac{T_3 L}{GJ} &= 1.45 \times 10^{-3}\,\mathrm{rad} + \frac{2\,\mathrm{N\,m} \cdot 40\,\mathrm{cm}}{1.03\,\mathrm{cm}^4 \cdot 25.9\,\mathrm{GPa}} \\ &= 1.45 \times 10^{-3}\,\mathrm{rad} + 3.00 \times 10^{-3}\,\mathrm{rad} \\ &= 4.45 \times 10^{-3}\,\mathrm{rad} = 0.26° \end{aligned}$$