Exercise 1

We consider a loaded beam with a cross-section in T-shape (fig. 1). The beam is characterized by the following values: $E = 20\,\text{GPa}$, $\nu = 0.2$, $t_1 = 2\,\text{cm}$, $t_2 = 1\,\text{cm}$, $\omega_1 = 1\,\text{cm}$, $\omega_2 = 4\,\text{cm}$, $L = 21\,\text{cm}$ and $F = 100\,\text{N}$. We define $z = 0$ at the top of the beam.

a) We will want to find the internal moment of the beam $M(x)$. Which axis should you consider for the calculation of the moment of area?

b) What is the distance z_c between the centroid and the top of the bar ? Calculate the moment of area at the centroid.

c) Determine the expression of $q(x)$. Deduce $V(x)$ and $M(x)$ by integration.

d) *Reminder: the formula for stress in a bent beam with a constant cross section is given by:* $\sigma_z(x) = \frac{M_y(x)}{I_y} \cdot (z - z_c)$*, where* z_c *is the position of the centroid.*
What is $\sigma_z(x)$ as a function of F and the geometrical dimensions ? Plot $\sigma_z(x)$ for the top of the beam $\sigma_{z=0cm}(x)$, the neutral axis $\sigma_{z=z_c}(x)$ and the bottom of the beam $\sigma_{z=3cm}(x)$.

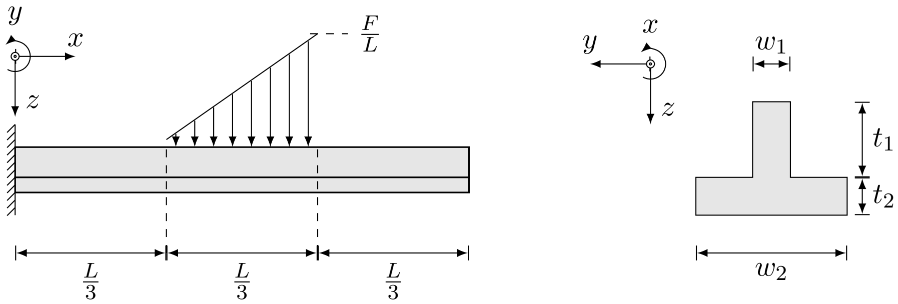


Figure 1: Schematic of the loaded bar and its cross-section.

Exercise solution 1

Given: Geometry, load.

Asked: Centroid, second moment of area I_y, $q(n)$, $V(n)$ and $(M(x)$; plots of $\sigma_{z=0}(x)$, $\sigma_{z=z_c}(x)$ and $\sigma_{z=3}(x)$.

Relevant relationships:

Steiner's Theorem

$$I_a = I_{cg} + Ad^2$$

Centroid Formula

$$z_c = \frac{\sum_i A_i z_{c,i}}{\sum_i A_i}$$

Stress in bent beam

$$\sigma_x(z) = \frac{M_y(x)}{I_y(x)} \cdot (z - z_c)$$

a)

The load on the cantilever will induce a moment in the y-direction. We thus need to find I_y.

b)

Finding the centroid is straight-forward:

$$z_c = \frac{\sum_i A_i z_{c,i}}{\sum_i A_i} = \frac{w_1 t_1 z_{c,1} + w_2 t_2 z_{c,2}}{w_1 t_1 + w_2 t_2} = 2\ cm$$

As given in the formula section, $I_y = \int_A z^2 dA$. It is however easier to look at the cross-section as an assembly of two rectangles and use Steiner's Theorem.

$$\begin{aligned}
I_{y,1} &= \frac{w_1 (t_1)^3}{12} = \frac{2}{3}\ cm^4 \\
I_{y,2} &= \frac{w_2 (t_2)^3}{12} = \frac{1}{3}\ cm^4 \\
I_y^{z=z_c} &= I_{y,1} + w_1 t_1 (z_c - z_{c,1})^2 + I_{y,2} + w_2 t_2 (z_c - z_{c,2})^2 \\
&= \frac{2}{3} + 1 \cdot 2 \cdot (2-1)^2 + \frac{1}{3} + 4 \cdot 1 \cdot (2 - 2.5)^2 = 4\ cm^4
\end{aligned}$$

c)

The load function here is composed of a linearly distributed load that is 0 at $x = \frac{1}{3}L$ and increases linearly to $\frac{F}{L}$ at $x = \frac{2}{3}L$. We write the load function as

$$q(x) = \frac{3F}{L^2}\left\langle x - \tfrac{L}{3} \right\rangle^1 - \frac{3F}{L^2}\left\langle x - \tfrac{2L}{3} \right\rangle^1 - \frac{F}{L}\left\langle x - \tfrac{2L}{3} \right\rangle^0$$

Integration yields

$$\begin{aligned}
V(x) &= -\int q(x)\,\mathrm{d}x + C_1 \\
&= -\frac{3F}{2L^2}\left\langle x - \tfrac{L}{3}\right\rangle^2 + \frac{3F}{2L^2}\left\langle x - \tfrac{2L}{3}\right\rangle^2 + \frac{F}{L}\left\langle x - \tfrac{2L}{3}\right\rangle^1 + C_1 \\
M(x) &= \int V(x)\,\mathrm{d}x + C_2 \\
&= -\frac{3F}{6L^2}\left\langle x - \tfrac{L}{3}\right\rangle^3 + \frac{3F}{6L^2}\left\langle x - \tfrac{2L}{3}\right\rangle^3 + \frac{F}{2L}\left\langle x - \tfrac{2L}{3}\right\rangle^2 + C_1 x + C_2
\end{aligned}$$

The constants are determined by the boundary conditions:

$$V(L) = 0 \rightarrow C_1 = \frac{F}{6}$$

and:

$$M(L) = 0 \quad \rightarrow \quad C_2 = -\frac{5FL}{54}$$

With substituted constants:

$$\begin{aligned}
V(x) &= -\int q(x)\,\mathrm{d}x + C_1 \\
&= -\frac{3F}{2L^2}\left\langle x - \tfrac{L}{3}\right\rangle^2 + \frac{3F}{2L^2}\left\langle x - \tfrac{2L}{3}\right\rangle^2 + \frac{F}{L}\left\langle x - \tfrac{2L}{3}\right\rangle^1 + \frac{F}{6} \\
M(x) &= \int V(x)\,\mathrm{d}x + C_2 \\
&= -\frac{3F}{6L^2}\left\langle x - \tfrac{L}{3}\right\rangle^3 + \frac{3F}{6L^2}\left\langle x - \tfrac{2L}{3}\right\rangle^3 + \frac{F}{2L}\left\langle x - \tfrac{2L}{3}\right\rangle^2 + \frac{F}{6}x - \frac{5FL}{54}
\end{aligned}$$

d)

To plot the stress in the x-direction, we use the calculated moment as well as the found second moment of area and the z position at which we want to know the stress:

$$\sigma_z(x) = \frac{M_y(x)}{I_y} \cdot (z - z_c) = \frac{M_y(x)\,[Nm]}{4 \cdot 10^{-8}\,[m^4]} \cdot (z - 2 \cdot 10^{-2}\,[m])$$

At $z = z_c$, we find:

$$\sigma_z(x) = \frac{M_y(x)}{I_y(x)} \cdot (z_c - z_c) = 0$$

We check that there is no stress in the neutral axis (which is its definition).
At $z = 0$, we find:

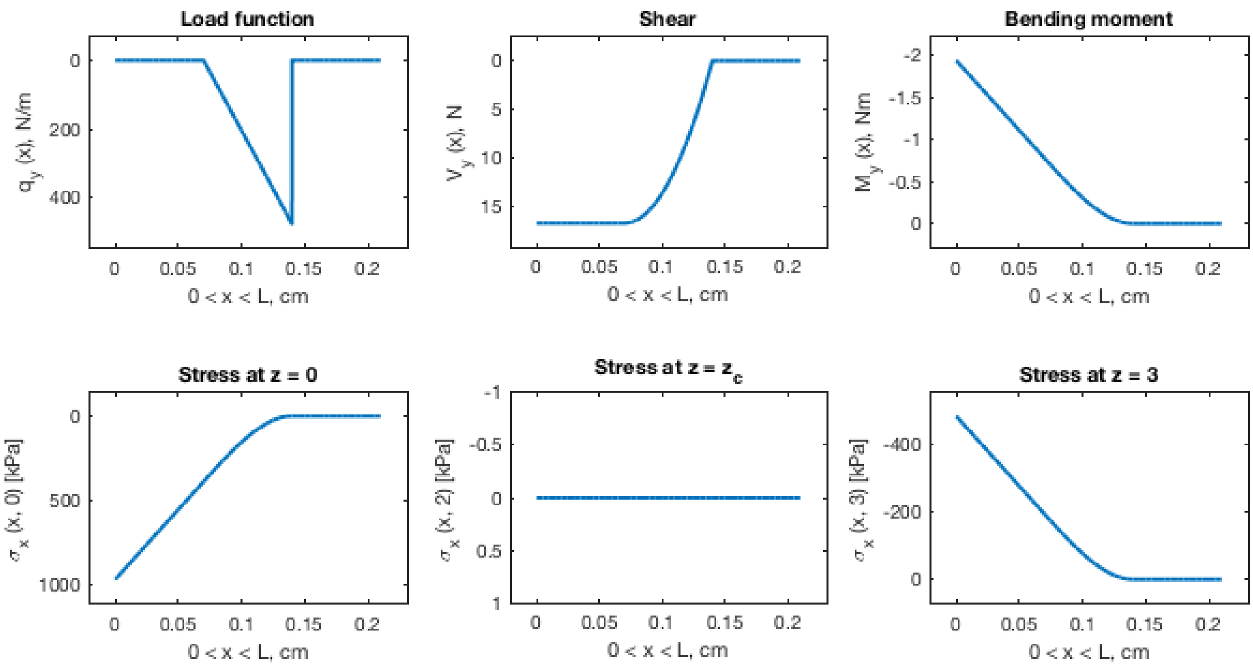


Figure 2: Graphs of the different calculated entities

$$\sigma_z(x) = \frac{M_y(x)}{I_y} \cdot (0 - z_c)$$

where

$$M_y(x) = -\frac{3F}{6L^2}\left\langle x - \tfrac{L}{3}\right\rangle^3 + \frac{3F}{6L^2}\left\langle x - \tfrac{2L}{3}\right\rangle^3 + \frac{F}{2L}\left\langle x - \tfrac{2L}{3}\right\rangle^2 + \frac{F}{6}x - \frac{5FL}{54}$$

Similarly at $z = 3$, we find:

$$\sigma_z(x) = \frac{M_y(x)}{I_y} \cdot (3 - z_c)$$

The moment as well as the stresses as a function of x are shown in fig. 2.

Exercise 2

a) The cross-section of an H-beam is shown in fig. 3. Find the centroid and second moments of area I_y and I_z of the beam.

b) For the two cross-sections shown in fig. 4, find the second moments of area I_y and I_z.

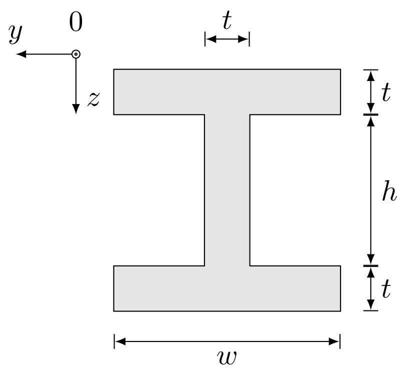


Figure 3: Cross-section of an H-beam.

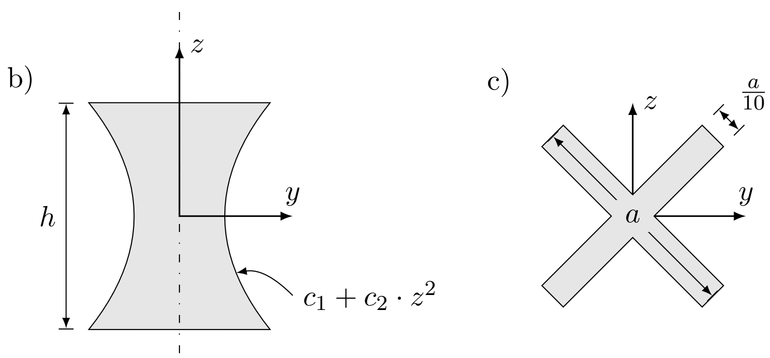


Figure 4: Cross-sections of more complicated beams.

Exercise solution 2

Given: Geometry.

Asked: Centroid location, second moments of area I_y and I_z.

Relevant relationships:

Second moment of area of a beam with rectangular cross section

$$I_y = \frac{h^3 \cdot w}{12} \qquad I_z = \frac{w^3 \cdot h}{12}$$

Parallel axis theorem for shifting elements with area A by a distance d

$$I_a = I_{a,c} + d^2 A$$

Second moment of area in y direction

$$I_y = \int_A z^2 dA$$

Second moment of area in z direction

$$I_z = \int_A y^2 dA$$

a)

Since the structure is symmetrical, the centroid is in the center of the structure (intersection of both symmetry–axes).

While the second moment of area can be calculated using integration, it is much easier to divide the cross section into rectangular elements of which the second moment of area is known and use the parallel axis theorem.

For I_z the parallel axis theorem is not even needed, as none of the elements are displaced from the symmetry axis in z direction. We find directly for the central crossbeam and two flanges

$$I_z = \frac{t^3 h}{12} + 2 \cdot \frac{w^3 t}{12} = \frac{t^3 h}{12} + \frac{w^3 t}{6}$$

In case of I_y we need the parallel axis theorem, as the center of mass of each flange does not lie on the symmetry axis in y-direcion. With the flange area $A = w \cdot t$ and the center–of–mass displacement $d = \frac{1}{2}(t + h)$ for two flanges we get

$$I_y = \frac{t \cdot h^3}{12} + 2 \cdot \left(\frac{t^3 w}{12} + w \cdot t \cdot \left(\frac{t+h}{2} \right)^2 \right) = \frac{t \cdot h^3}{12} + \frac{2t^3 w}{3} + t^2 w \cdot h + \frac{t \cdot w \cdot h^2}{2}$$

b)

As given in the formula section, $I_y = \int_A z^2 dA$. It becomes, for the given geometry,

$$I_y = \int_{-h/2}^{h/2} \int_{-(c_1+c_2z^2)}^{c_1+c_2z^2} z^2 \, \mathrm{d}y \, \mathrm{d}z$$

$$I_y = 4 \int_0^{h/2} \int_0^{c_1+c_2z^2} z^2 \, \mathrm{d}y \, \mathrm{d}z$$

$$I_y = 4 \int_0^{h/2} z^2 \left(c_1 + c_2 z^2 \right) \mathrm{d}z$$

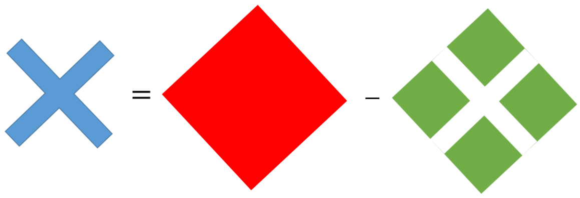
$$I_y = \frac{c_1}{6} h^3 + \frac{c_2}{40} h^5$$

It is the same method for I_z :

$$I_z = 4 \int_0^{h/2} \int_0^{c_1+c_2z^2} y^2 \,\mathrm{d}y \,\mathrm{d}z$$
$$I_z = 4 \int_0^{h/2} \frac{(c_1+c_2z^2)^3}{3} \,\mathrm{d}z$$
$$I_z = \frac{1}{672}c_2^3h^7 + \frac{1}{40}c_1c_2^2h^5 + \frac{1}{6}c_1^2c_2h^3 + \frac{2}{3}c_1^3h$$

c)

Moments of area for the y and z directions are equal for symmetry reasons. The cross is equivalent to a big square at 45^o with a side length of a, minus 4 smaller squares with a side of $\frac{9}{20}a$.



The moment of area is, for the red square :

$$I_1 = 4 \int_0^{a\frac{\sqrt{2}}{2}} \int_0^{a\frac{\sqrt{2}}{2}-z} z^2 \,\mathrm{d}y \,\mathrm{d}z$$
$$I_1 = \frac{1}{12}a^4$$

The green squares have a width of $b = \frac{a-\frac{1}{10}a}{2} = \frac{9}{20}a$. It is possible to use the previous calculation if we replace a by b to get the y moment of area of each of the two green squares along the y axis :

$$I_2 = \frac{1}{12}b^4$$
$$I_2 = \frac{1}{12}\left(\frac{9}{20}a\right)^4$$
$$I_2 = \frac{2187}{640000}a^4$$

For the top and bottom green squares, we need to take into account the distance between the centroid and the y axis. This distance is

$$d = a\frac{\sqrt{2}}{2} - \frac{9}{20}a\frac{\sqrt{2}}{2}$$
$$d = \frac{11}{20}a\frac{\sqrt{2}}{2}$$

The y moment of area of each of the top and bottom green square is :

$$I_3 = \frac{1}{12}b^4 + d^2b^2$$
$$I_3 = \frac{21789}{640000}a^4$$

Finally, the moments of area of the cross are

$$I_y = I_1 - 2I_2 - 2I_3$$
$$I_y = \frac{1009}{120000}a^4$$
$$I_z = \frac{1009}{120000}a^4$$

Exercise 3

Occasionally, beams are made by joining two different materials, for example for a bimetal sensing element or actuator. Fig. 5 shows such a beam and its corresponding cross section.
Assume the top material is aluminium with a Young's modulus $E_{\mathrm{Al}} = 70\,\mathrm{GPa}$ and a thickness $t_1 = 10\,\mathrm{mm}$. The bottom material is copper with a Young's modulus $E_{\mathrm{Cu}} = 120\,\mathrm{GPa}$ and a thickness $t_2 = 7\,\mathrm{mm}$. The beam has a width $w = 20\,\mathrm{mm}$.

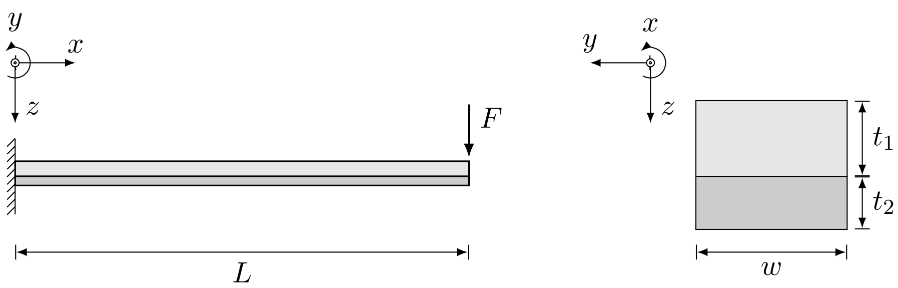


Figure 5: Composite beam under bending.

a) Find an expression for the neutral axis when the Young's modulus in z-direction is variable. Find the neutral axis of the given beam.

b) Find the equivalent flexural rigidity (EI) of the beam.

Exercise solution 3

Given:

- Geometry $t_1 = 10\,\mathrm{mm}$, $t_2 = 7\,\mathrm{mm}$, $w = 20\,\mathrm{mm}$.
- Young's moduli $E_{\mathrm{Al}} = 70\,\mathrm{GPa}$ and $E_{\mathrm{Cu}} = 120\,\mathrm{GPa}$.

Asked: Centroid location and equivalent flexural rigidity.

Relevant relationships:

- Second moment of area of a square

$$I_y = \frac{h^3 \cdot w}{12}$$

- Normal strain due to bending

$$\varepsilon_x = \kappa z$$

- Normal stress

$$\sigma_x = \varepsilon_x \cdot E$$

a)

To find the neutral axis, we can not use the centroid formula as with a uniform beam, since different parts of the beam will cause different amounts of normal stress in a section. However the shear strain is linear with the distance from the neutral axis. Figure 6 shows an element of the beam.

Figure 6: Composite beam element.

The equilibrium of forces in the x-direction tells us that

$$\begin{aligned} 0 &= \iint_A \sigma_x(x,z)\,\mathrm{d}A = \int_0^w \int_0^{t_1+t_2} E(z)\varepsilon_x \,\mathrm{d}z\,\mathrm{d}y \\ &= \int_0^w \int_0^t E(z)\kappa(z-z_c)\,\mathrm{d}z\,\mathrm{d}y = \kappa \int_0^w \int_0^t E(z)\cdot(z-z_c)\,\mathrm{d}z\,\mathrm{d}y \end{aligned}$$

where we can split up the integral

$$0 = \int_0^w \int_0^t E(z)\cdot(z-z_c)\,\mathrm{d}z\,\mathrm{d}y = \int_0^w \int_0^t E(z)\cdot z\,\mathrm{d}z\,\mathrm{d}y - \int_0^w \int_0^t E(z)\cdot z_c\,\mathrm{d}z\,\mathrm{d}y$$

$$\int_0^w \int_0^t E(z)\cdot z\,\mathrm{d}z\,\mathrm{d}y = z_c \int_0^w \int_0^t E(z)\,\mathrm{d}z\,\mathrm{d}y$$

so in analogy to the normal centroid formula we get a centroid formula for varying Young's modulus

$$z_c = \frac{\iint_A E(z)\cdot z\,\mathrm{d}A}{\iint_A E(z)\,\mathrm{d}A} = \frac{\sum_i E_i A_i \cdot z_{c,i}}{\sum E_i A_i}$$

Which gives us for the given beam (z from the top of the beam downwards)

$$\begin{aligned} z_c &= \frac{E_{\mathrm{Al}} \cdot z_{c,\mathrm{Al}} \cdot t_1 + E_{\mathrm{Cu}} \cdot z_{c,\mathrm{Cu}} \cdot t_2}{E_{\mathrm{Al}} \cdot t_1 + E_{\mathrm{Cu}} \cdot t_2} \\ &= \frac{70\,\mathrm{GPa} \cdot 5\,\mathrm{mm} \cdot 10\,\mathrm{mm} + 120\,\mathrm{GPa} \cdot 13.5\,\mathrm{mm} \cdot 7\,\mathrm{mm}}{70\,\mathrm{GPa} \cdot 10\,\mathrm{mm} + 120\,\mathrm{GPa} \cdot 7\,\mathrm{mm}} = 9.64\,\mathrm{mm} \end{aligned}$$

so slightly above the junction.

b)

With the neutral axis the equivalent flexural rigidity is simply given by the individual components with respect to the neutral axis. In this case, using the parallel axis theorem

$$\begin{aligned} EI_{\mathrm{equiv}} &= E_{\mathrm{Al}} \cdot \left(I_{\mathrm{Al}} + A_{\mathrm{Al}} \cdot d_{\mathrm{Al}}^2\right) + E_{\mathrm{Cu}} \cdot \left(I_{\mathrm{Cu}} + A_{\mathrm{Cu}} \cdot d_{\mathrm{Cu}}^2\right) \\ &= w \cdot \left(E_{\mathrm{Al}} \cdot \left(\frac{t_1^3}{12} + t_1 \cdot (z_{c,\mathrm{Al}} - z_c)^2\right) + E_{\mathrm{Cu}} \cdot \left(\frac{t_2^3}{12} + t_2 \cdot (z_c - z_{c,\mathrm{Cu}})^2\right)\right) \\ &= 737\,\mathrm{N\,m^2} \end{aligned}$$