Homework

Exercise 1. Let K be a normed (non-Archimedean) field. Let $K\langle x_1, \ldots, x_n \rangle$ be a Tate algebra in n variables over K (i.e. the algebra of power series which are convergent on the closed unit disc). Define a function $|\cdot|$ from $K\langle x_1, \ldots, x_n \rangle$ to $\mathbb{R}_{\geq 0}$ by

$$|\cdot| \colon K\langle x_1, \ldots, x_n \rangle \to \mathbb{R}_{\geq 0}$$
$$\sum_{\mu \in \mathbb{N}^n} a_\mu x^\mu \mapsto \max_{\mu \in \mathbb{N}^n} |a_\mu|.$$

Show that $(K\langle x_1, \ldots, x_n \rangle, |\cdot|)$ is a normed ring.

Exercise 2. For $f \in K\langle x_1, \ldots, x_n \rangle$, is it true that $|f| = \max\{|f(c)| \mid c \in \mathbb{D}_n\}$? Maybe by putting a condition on K?

Exercise 3. Let A be a non-Archimedean topological ring, denote by A° the set of power-bounded elements, and by $A^{\circ\circ}$ the set of topologically nilpotent elements. Prove that

(i) $A^{\circ\circ} \subseteq A^\circ$;

(ii) $A^\circ \subseteq A$ is a subring (is it open?);

(iii) $A^{\circ\circ} \subseteq A^\circ$ is an ideal.

Exercise 4. Let K be a complete normed field with valuation ring $(R, \mathfrak{m})$, and take $f \in K\langle x_1, \ldots, x_n \rangle$ with $|f| = 1$. Then the following are equivalent:

(i) $f \in K\langle x_1, \ldots, x_n \rangle^\times$

(ii) if $\overline{f} \in (R/\mathfrak{m})[x_1, \ldots, x_n]$ is the reduction of f in the residue field, then $\overline{f} \in (R/\mathfrak{m})^\times$.

(iii) The only coefficient of f having norm 1 is the constant coefficient.

Exercise 5. Let K be a complete normed field and consider the Tate algebra $T \coloneqq K\langle x_1, \ldots, x_n \rangle$. Let $g \in T$ be x_n–distinguished of order s, and let $f \in T$ be arbitrary. By Weierstrass division, there exist $q, r \in T$ such that $f = qg + r$ and r is a polynomial in the x_n–variable of degree $< s$. Prove that q, r are unique with this property, and that $|f| = \max\{|q||g|, |r|\}$.

Exercise 6. Let K be a complete normed field. Recall that we denote by $\overline{\mathbb{D}}_n \subseteq \overline{K}^n$ the closed unit disc in $\overline{K}^n$. Also recall that for every $(\lambda_1, \ldots, \lambda_n) \in \overline{\mathbb{D}}_n$, we have a well-defined evaluation morphism

$$\begin{aligned} \mathrm{ev}_{(\lambda_1,\ldots,\lambda_n)} \colon K\langle x_1, \ldots, x_n\rangle &\to \overline{K} \\ f &\mapsto f(\lambda_1, \ldots, \lambda_n). \end{aligned}$$

Show that the map

$$\begin{aligned} \overline{\mathbb{D}}_n &\to \operatorname{Specm} K\langle x_1, \ldots, x_n\rangle \\ (\lambda_1, \ldots, \lambda_n) &\mapsto \ker \mathrm{ev}_{(\lambda_1,\ldots,\lambda_n)} \end{aligned}$$

is surjective. Furthermore, the fibers are given by Galois orbits.

Exercise 7. Let $\mathbb{Q}_p^{\mathrm{nr}} = \mathbb{Q}_p\big(x \in \overline{\mathbb{Q}_p} \mid x^n = 1 \text{ for some } n \text{ prime to } p\big)$ be the field obtained by adjoining all the prime-to-p roots of unity to $\mathbb{Q}_p$. Show that the p–adic norm $|\cdot|_p$ extends uniquely both to $\mathbb{Q}_p^{\mathrm{nr}}$ and to $\mathbb{C}_p \coloneqq \overline{\mathbb{Q}_p}^{\wedge_p}$.

Exercise 8. Let K be $\mathbb{Q}_p$ or $\mathbb{Q}_p^{\mathrm{nr}}$. Let $|\cdot| \in \operatorname{Spa}(K\langle x\rangle, A\langle x\rangle)$ be a valuation with support $\operatorname{Ker} \mathrm{ev}_\lambda$ for some $\lambda \in \overline{K}$. Then show that the valuation $|\cdot|$ induced on $K(\lambda)$ is continuous.

Exercise 9. Let Γ be a totally ordered abelian group, such that for all $a, b \in \Gamma$ with $a > 0$ and $b \geq 0$, there exists $n \in \mathbb{Z}_{>0}$ such that $b < na$ (or equivalently $\operatorname{rank} \Gamma = 1$). Show that there exists an injective homomorphism of totally ordered abelian groups $\varphi\colon \Gamma \hookrightarrow \mathbb{R}$.

Exercise 10. Let A be an f-adic ring, and let $|\cdot|\colon A \to \Gamma$ be some valuation. Recall that the characteristic subgroup $c\Gamma_{|\cdot|}$ is by definition the convex subgroup of Γ generated by all elements $\gamma \geq 1_\Gamma$ in the image of $|\cdot|$. Show that

$$c\Gamma_{|\cdot|} = \{\gamma \in \Gamma \mid \exists m, n \in \mathbb{Z} \text{ with } m < 0 < n \ \exists a \in A: \ |a| \geq 1 \text{ and } |a|^m \leq \gamma \leq |a|^n\}.$$

Exercise 11. Let A be f-adic, $|\cdot|\colon A \to \Gamma$ a valuation and $H \leq \Gamma$ a convex subgroup. Then the following are equivalent:

(i) The function $|\cdot|_H\colon A \to H$ defined by

$$\begin{aligned} |\cdot|_H \colon A &\to H \\ a &\mapsto |a|_H \coloneqq \begin{cases} |a| & \text{if } |a| \in H, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

is a valuation.

(ii) $H \supseteq c\Gamma_{|\cdot|}$

(iii) For the composition

$$\begin{tikzcd} A \arrow[r, "|\cdot|"] \arrow[rr, bend right, "|\cdot|^*"'] & \Gamma \arrow[r] & \Gamma/H \end{tikzcd}$$

we have $|a|^* \leq 1$ for all $a \in A$.

(iv) If for some $a \in A$ we have $|a| \notin H$ then $|a| < 1$

Exercise 12. Let X be a quasi-compact and quasi-separated space. A subset $Y \subseteq X$ is said to be constructible if it is of the form $Y = U \cap (X \setminus V)$ for quasi-compact open subsets $U, V \subseteq X$. If X is the topological space underlying a scheme, then does this agree with the scheme theoretic notion of constructibility?

Exercise 13. Give an example of an f-adic ring A such that $\mathrm{Cont}(A) \subseteq \mathrm{Spv}(A)$ is not closed.

Exercise 14. Let $f\colon A \to B$ be an adic morphism of f-adic rings, let A_0, B_0 be rings of definition of A resp. B such that $f(A_0) \subseteq B_0$. Show that for any ideal of definition $I \subseteq A_0$, the extension $f(I) \cdot B_0$ of I under f is an ideal of definition for B.

Exercise 15. Consider the ring

$$\widehat{\mathbb{Z}} := \varprojlim_{n \in \mathbb{Z}_{>0}} \mathbb{Z}/n\mathbb{Z}.$$

Show that $\widehat{\mathbb{Z}}$ is not f-adic.

Exercise 16. In $\mathbb{Q}_p\langle x \rangle$, exhibit all the points that lie in the intersection

$$R\left(\frac{x, p}{p}\right) \cap R\left(\frac{x-1, p}{p}\right)$$

Exercise 17. Let (A, A^+) be an affionoid ring, and $J \subseteq A$ an ideal. Let $(A/J)^+$ be the integral closure of A^+ in A/J. Then

$$\mathrm{Spa}\left(A/J, (A/J)^+\right) \cong \mathrm{Spa}(A, A^+) \cap \mathrm{Supp}^{-1}(V(J)).$$

In particular, $(A/J, (A/J)^+)$ is affinoid.