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THE TOPOLOGY OF COMPLEX PROJECTIVE VARIETIES AFTER S. LEFSCHETZ

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AFTER THE topology of complex algebraic curves, i.e. the genus of Riemannian surfaces, had been understood mathematicians like Picard[12] and Poincaré[12*a*] went on to the next dimension and began to investigate the topology of complex algebraic surfaces. From 1915 on Lefschetz continued their work and extended it to higher dimensional varieties. In 1924 he published his famous exposition [L] of this work.

When it was written knowledge of topology was still primitive and Lefschetz "made use most uncritically of early topology à la Poincaré and even of his own later developments"†. This makes it nowadays rather difficult to understand the topological parts of [L] properly. But that is not the only difficulty: Implicitly Lefschetz quite often appeals to geometric intuition where we would like to see a more precise argument.

Thus there is some temptation to discard Lefschetz's original "proofs" and adopt instead the more recent methods which have been employed to obtain many of his results, using Hodge's theory of harmonic differential forms or Morse theory or sheaf theory and spectral sequences. But none of these very elegant methods yields Lefschetz's full geometric insight, e.g. they do not show us the famous "vanishing cycles".

The first attempt to rewrite the topological part of [L] using modern singular homology theory was made more than twenty years ago by Wallace[16]. But the details of his presentation are too complicated to popularize Lefschetz's original methods. Wallace leaves the realm of algebraic geometry far too early when he makes Lefschetz's intuitive arguments precise. Furthermore he does not give a complete picture of Lefschetz's achievements.

In the following I make a new attempt to present Lefschetz's almost sixty year old investigations rigorously but as geometrically as he did in [L]. For topologists Lefschetz is usually interesting for the work he did in pure topology after he had completed [L]. But [L] has at least "a unique historical interest in being almost the first account of the topology of a construct of importance in general mathematics which is not trivial" (Hodge). We may furthermore speculate how much of the contributions of Poincaré, Lefschetz and others to algebraic topology we owe to the difficulties they encountered with the topology of algebraic varieties.

The necessary prerequisites in algebraic geometry can be found in the first two chapters of Shafarevich's book[13]. The main tool from differential topology is Ehresmann's fibration theorem, which for the convenience of the reader is stated in 3.0. (Strangely enough this theorem is not included in the standard textbooks.) As far as homology theory is concerned a textbook like Dold's[6] will amply suffice. Furthermore some basic facts about the fundamental group and the homotopy lifting theorem for fibre bundles will be used.

†S. Lefschetz in his autobiography, *Bull. Am. Math. Soc.* **74**, (1968) 854–879.

A descriptive outline may be obtained by reading §1, 3, 4, 6.1–3., 7.1–3. and 8.1. I must admit that I have not succeeded in proving the "Hard Lefschetz Theorem" rigorously by purely topological methods. I have merely collected a lot of equivalent formulations of it and some consequences in §4 and §7. In [L] the Picard–Lefschetz formula and the Hard Lefschetz Theorem are the two fundamental facts upon which the further investigations are built. I have not kept to the original order of [L] because many of Lefschetz's results do not require the full strength of these two theorems. They follow already from a less deep result, which I call the Fundamental Lemma, (3.2.2) below.

Transcendental analytical methods play an essential rôle in complex algebraic geometry: see e.g. [5, 5*a*]. But in the following exposition I want to emphasize the directness of Lefschetz's methods, i.e. to investigate the topology of a variety as far as possible by geometrical and topological methods before embarking on transcendental considerations.

§1. THE MODIFICATION OF A PROJECTIVE VARIETY WITH RESPECT TO A PENCIL OF HYPERPLANES

1.1 Let $\mathbf{P}_N$ denote N-dimensional complex projective space. A *pencil* in $\mathbf{P}_N$ consists of all hyperplanes which contain a fixed $(N-2)$-dimensional projective subspace A, which is called the *axis* of the pencil.

The hyperplanes of $\mathbf{P}_N$ are the points of the dual projective space $\check{\mathbf{P}}_N$. The following notation will be used

$$\mathbf{P}_N \supset H_y \qquad y \in \check{\mathbf{P}}_N.$$

The hyperplanes $\{H_t\}$ form a pencil if and only if the corresponding points $\{t\}$ form a projective line $G \subset \check{\mathbf{P}}_N$. Hence the pencil is denoted by $\{H_t\}_{t\in G}$.

1.2. The main object under consideration is a closed, irreducible subvariety $X \subset \mathbf{P}_N$, without singularities. Let

$$\dim X = n.$$

(Lefschetz actually studies a hypersurface $X \subset \mathbf{P}_{n+1}$ which has singularities, but only those occurring in a generic projection of a smooth variety $Y \subset \mathbf{P}_N$ into $\mathbf{P}_{n+1}$, see [L] Chap. V, §1.)

The variety X is intersected by a pencil $\{H_t\}_{t\in G}$ of hyperplanes,

$$X_t = X \cap H_t, \qquad t \in G$$

so that

$$X = \bigcup_{t\in G} X_t$$

is the union of the *hyperplane sections* X_t. Off the *exceptional subset*

$$X' = X \cap A$$

X can be looked at as a fibration over G with fibres $X_t \backslash X'$. This is an important fact for Picard's and Lefschetz's geometric arguments. In order to make their arguments precise and at the same time easier to understand it is very convenient to modify (blow up) X along X' to get a new variety Y with a map $f: Y \to G$ such that the fibres $f^{-1}(t)$ are the whole hyperplane sections X_t. This idea can be found in Wallace's book [16]. But at this stage he leaves the realm of algebraic geometry and constructs Y by complicated topological cutting and pasting. It is much easier and better to stay in the realm of algebraic geometry and to define the *modification*

$$Y = \{(x, t) \in X \times G \mid x \in H_t\}.$$

Then there are two projections

$$X \xleftarrow{p} Y \xrightarrow{f} G.$$

Let

$$Y' = p^{-1}(X') = X' \times G \tag{1.2.1}$$

denote the exceptional set. The complement is mapped isomorphically

$$p: Y \backslash Y' \simeq X \backslash X'. \tag{1.2.2}$$

and each fibre of f is mapped isomorphically onto the corresponding hyperplane section,

$$p: Y_t = f^{-1}(t) \simeq X_t, \qquad t \in G. \tag{1.2.3}$$

1.3. Lefschetz in [L] studies not only pencils $\{X_t\}_{t\in G}$ of hyperplane sections but more generally linear (i.e. one parameter) systems of *hypersurfaces* of X. He states (e.g. in [L] Chap. IV, §2) that the restriction to hyperplane sections does not diminish the generality. This is justified by the *Veronese embedding* of projective spaces (see e.g. [13, Chap. I, §4, §4.2]): Consider $\mathbf{P}_N$ with its homogeneous coordinates $(x_0: \ldots : x_N)$. Let $\mu_0, \ldots, \mu_M$ denote all monomials of degree d in $x_0, \ldots, x_N$. Thus $M = \binom{N+d}{d} - 1$. The Veronese embedding of degree d is defined to be

$$v: \mathbf{P}_N \to \mathbf{P}_M, \quad (x_0: \ldots : x_N) \mapsto (\mu_0: \ldots : \mu_M).$$

It is a regular embedding of $\mathbf{P}_N$ onto the Veronese variety $v(\mathbf{P}_N) \subset \mathbf{P}_M$. There is a one-to-one correspondence between the hypersurfaces F of degree d in $\mathbf{P}_N$ and the hyperplanes of $\mathbf{P}_M$: If F is given by the homogeneous polynomial equation $f(x_0, \ldots, x_N) = \sum_{j=0}^{M} a_j\mu_j = 0$, the corresponding hyperplane $H_F \subset \mathbf{P}_M$ is given by $\sum_{j=0}^{M} a_j y_j = 0$. The image $v(F)$ is the intersection

$$v(F) = v(\mathbf{P}_N) \cap H_F.$$

The point $x \in F$ is simple if and only if H_F intersects $v(\mathbf{P}_N)$ at $v(x)$ transversally. Consider now an arbitrary Zariski-closed subset $X \subset \mathbf{P}_N$ and let $x \in X \cap F$ be a simple point of both X and F. If F intersects X at x transversally, H_F intersects $v(X)$ at $v(x)$ transversally.

1.4. Let us now return to pencils of hyperplanes. In general some (finitely many) hyperplanes of a fixed pencil are tangent to X and the points of tangency become singular points of the corresponding hyperplane sections. Lefschetz admits only pencils for which at worst simple singularities occur (see [L], Chap. II, §8 and Chap. V, §2). These pencils are called Lefschetz pencils in the Seminaire Géométrie Algébrique: see [22, Exposé XVII]. They will now be described using the notion of transversality. At the same time it will become clear that they are generic. The following treatment is similar to [22, Exposé XVII] but may be easier to understand for those who are less trained in modern algebraic geometry and want only to look at the classical case of complex projective varieties.

(1.4.1) *All hyperplanes of* $\mathbf{P}_N$ *which are tangent to* X *form a closed irreducible subvariety* $\check{X} \subset \check{\mathbf{P}}_N$ *of at most* $N-1$ *dimensions.* It is called the *dual variety* of X.

This will be proved in 2.1. In general $\check{X}$ has singularities even if X is smooth, and $\dim \check{X} = N-1$ even if $\dim X < N-1$. The following corollary is almost equivalent to (1.4.1):

(1.4.2) *The hyperplanes which intersect* X *transversally form the non-empty Zariski-open subset* $\check{\mathbf{P}}_N \backslash \check{X}$ *of* $\check{\mathbf{P}}_N$.

1.5. If $\check{X}$ is a hypersurface it has a degree $r > 0$. This degree is called the *class* of X. (This agrees with the usual definition for plane curves.) If $\dim \check{X} \le N-2$ the class of X is 0 by definition.

Let $b \in \check{\mathbf{P}}_N \backslash \check{X}$ (so that H_b intersects X transversally). All projective lines in $\check{\mathbf{P}}_N$ through b form an $(N-1)$-dimensional projective space E. *If class* $X = 0$ (*i.e.* $\dim \check{X} \le N-2$) *the lines which do not meet* $\check{X}$ *form a non-empty open subset in* E. *If class* $X = r > 0$ (*i.e.* $\dim \check{X} = N-1$) *the lines which avoid the singular set of* $\check{X}$ *and intersect* $\check{X}$ *transversally form a non-empty open subset in* E. *For each line* G *in this subset the intersection* $G \cap \check{X}$ *consists of* $r =$ *class* X *many points.*

In order to prove this result consider the projection with center b

$$\rho\colon \check{X} \to E, \quad \rho(y) = \textit{line through } b \textit{ and } y.$$

It is a regular map. Therefore $\rho(\check{X})$ is a closed subset of E with $\dim \rho(\check{X}) \le \dim \check{X}$. If $\dim \check{X} \le N-2$ the lines which do not meet $\check{X}$ form the non-empty open subset $E \backslash \rho(\check{X})$. If $\dim \check{X} = N-1$ the subset $C \subset \check{X}$ consisting of all singular points of $\check{X}$ together with the simple points y of $\check{X}$ where the line $\rho(y)$ is not transversal to $\check{X}$ (i.e. where ρ fails to have maximal rank $N-1$) is proper and closed, hence $\dim C \le N-2$ because $\check{X}$ is irreducible. Therefore $\rho(C)$ is a closed subset of at most $N-2$ dimensions, and the lines which intersect $\check{X}$ transversally form the non-empty open subset $E \backslash \rho(C)$.

1.6. Let $G \subset \check{\mathbf{P}}_N$ be a projective line which intersects $\check{X}$ transversally and avoids the singular set, so that in particular $G \cap \check{X} = \emptyset$ if $\dim \check{X} \le N-2$. Let $\{H_t\}_{t \in G}$ denote the corresponding pencil of hyperplanes in $\mathbf{P}_N$ with axis A.

(1.6.1) *The axis* A *intersects* X *transversally. Therefore the exceptional subsets* $X' = X \cap A$ *and* $Y' = p^{-1}(X') = X' \times G$ *are non-singular and have* $n-2$ resp. $n-1$ *dimensions.*

(1.6.2) *The modification Y of X along X' is irreducible and non-singular.*

(1.6.3) *The projection $f\colon Y \to G$ has $r = \text{class } X$ critical values, namely the points of $\check{X} \cap G$. There are the same number of critical points, i.e. no two lie in the same fibre of f.*

(1.6.4) *Every critical point is non-degenerate, i.e. with respect to local holomorphic coordinates the Hessian matrix of the second derivatives of f has maximal rank n at the critical point.*

These results will be proved in §2.5 and §2.6. The topological investigations begin in the third section. In order to understand them the following §2 can be omitted.

§2. THE DUAL VARIETY

2.1. Let $X \subset \mathbf{P}_N$ denote a closed irreducible subvariety of n dimensions which may have singularities, and let $X_e \subset X$ denote the non-empty open subset of its simple points. Define

$$V'_X = \{(x, y) \in \mathbf{P}_N \times \check{\mathbf{P}}_N \mid x \in X_e \quad \textit{and } H_y \textit{ is tangent to } X \textit{ at } x\}.$$

This is a quasi-projective subset of $\mathbf{P}_N \times \check{\mathbf{P}}_N$, because the set $\tilde{V} = \{(x, y) \in \mathbf{P}_N \times \check{\mathbf{P}}_N \mid x \in X,\ x \textit{ is singular or } H_y \textit{ is tangent to } X \textit{ at } x\}$ is closed in $\mathbf{P}_N \times \check{\mathbf{P}}_N$ and V'_X is open in $\tilde{V}$. The first projection

$$\pi_1\colon V'_X \to X_e, \qquad (x, y) \mapsto x$$

fibres V'_X locally trivially. The fibres are (isomorphic to) $(N - n - 1)$-dimensional projective subspaces of $\check{\mathbf{P}}_N$, in particular: If X is an hypersurface ($n = N - 1$), π_1 is an isomorphism. Hence V'_X is irreducible and has $N - 1$ dimensions. The same holds true for the closure V_X of V'_X in $\mathbf{P}_N \times \check{\mathbf{P}}_N$. It is called the *tangent hyperplane bundle* of X. The first projection maps V_X *onto* X,

$$\pi_1\colon V_X \to X, \quad (x, y) \mapsto x.$$

Consider now the second projection

$$\pi_2\colon V_X \to \check{\mathbf{P}}_N, \quad (x, y) \mapsto y.$$

Its image $\check{X} = \pi_2(V_X)$ is a closed irreducible subvariety of $\check{\mathbf{P}}_N$ of at most $N - 1$ dimensions, the so called *dual variety* of X. This definition of $\check{X}$ coincides with the definition of §1 when X has no singularities. In general $\check{X}$ has singularities even if X does not. The reason why the dual variety has been defined for singular varieties too is the following;

2.2. Duality Theorem. *The tangent hyperplane bundles of X and $\check{X}$ coincide*

$$V_{\check{X}} = V_X \textit{ and hence } \check{\check{X}} = X.$$

2.3. In order to prove this theorem and also the results of §1 the bundle

$$W = \{(x, y) \in \mathbf{P}_N \times \check{\mathbf{P}}_N \mid x \in X \cap H_y\} \tag{2.3.1}$$

of *all* hyperplane sections of X will be used. By the first projection

$$p_1\colon W \to X, \quad p_1(x, y) = x,$$

W is locally trivially fibred over X with the hyperplanes of $\check{\mathbf{P}}_N$ as fibres. For an explicit trivialization see (2.6.3) below. Hence W is closed in $\mathbf{P}_N \times \check{\mathbf{P}}_N$, irreducible, and has $N + n - 1$ dimensions. Obviously $V_X \subset W$ and $\pi_1 = p_1 | V_X$. The open set of simple points is $W_e = p_1^{-1}(X_e)$. At a simple point $(c, b) \in W$ the second projection

$$p_2\colon W \to \check{\mathbf{P}}_N, \quad p_2(x, y) = y$$

has maximal rank ($= N$) if and only if H_b intersects X at c transversally, in other words *V_X' is the set of simple points of W which are critical with respect to p_2.*

Before using W for the announced proofs another easy but important application will be made. For this assume $X \subset \mathbf{P}_N$ to be smooth. Remove the dual variety $\check{X}$ and its inverse image $p_2^{-1}(\check{X})$. Then $p_2\colon W \backslash p_2^{-1}(\check{X}) \to \check{\mathbf{P}}_N \backslash \check{X}$ is a proper mapping which everywhere has maximal rank $= N$. Therefore according to Ehresmann's fibration theorem (see §3.0 below) $W \backslash p_2^{-1}(\check{X})$ is a C^∞ locally trivial fibre bundle over $\check{\mathbf{P}}_N \backslash \check{X}$. Since $\check{\mathbf{P}}_N \backslash \check{X}$ is path-connected all fibres of $W \backslash p_2^{-1}(\check{X})$, i.e. *all transversal hyperplane sections X_y of X are diffeomorphic to one another.* If this is applied to the Veronese variety $X = v(\mathbf{P}_N) \subset \mathbf{P}_M$ of degree d (see 1.3.) we get the remarkable result:

(2.3.2) *All smooth hyperfaces of $\mathbf{P}_N$ which have the same degree d are diffeomorphic to one another.*

2.4. *Proof of the Duality Theorem* 2.2. Consider the subset $U \subset V_X$ consisting of all points (c, b) such that $c \in X_e$, $b \in \check{X}_e$, and $\pi_2 = p_2 | V_X$ has maximal rank ($= \dim \check{X}$) at (c, b). This set is open in V_X and non-empty. It is sufficient to prove that $U \subset V_{\check{X}}$ because this implies $V_X \subset V_{\check{X}}$. Since $\dim V_X = \dim V_{\check{X}}$ and $\check{X}$ and hence $V_{\check{X}}$ is irreducible, $V_X = V_{\check{X}}$. In order to prove $U \subset V_{\check{X}}$ let $(c, b) \in U$. The definition of W implies $\{c\} \times {}_cH \subset W$. Here ${}_cH \subset \check{\mathbf{P}}_N$ is the hyperplane of $\check{\mathbf{P}}_N$ which corresponds to $c \in \mathbf{P}_N$. Therefore $T_{(c,b)}(\{c\} \times {}_cH) \subset T_{(c,b)}W$ (here T_a means the tangent space at a) and

$$(Tp_2)(T_{(c,b)}(\{c\} \times {}_cH)) \subset (Tp_2)(T_{(c,b)}W).$$

The projection p_2 maps $\{c\} \times {}_cH$ isomorphically onto ${}_cH$, hence

$$(Tp_2)(T_{(c,b)}(\{c\} \times {}_cH)) = T_b({}_cH).$$

At (c, b) the rank of p_2 is $< N$. The preceding formulas show: The rank is $= N - 1$, more precisely

$$(Tp_2)(T_{(c,b)}W) = T_b({}_cH).$$

On the other hand $V_X \subset W$ implies $T_{(c,b)}V_X \subset T_{(c,b)}W$, hence

(2.4.1) $$(T\pi_2)(T_{(c,b)}V_X) \subset (Tp_2)(T_{(c,b)}W) = T_b({}_cH).$$

Since $\pi_2 = p_2 | V_X$ has maximal rank ($= \dim \check{X}$) at (c, b) and $b \in \check{X}$ is simple,

(2.4.2) $$T_b\check{X} = (T\pi_2)(T_{(c,b)}V_X) \subset T_b({}_cH),$$

i.e. ${}_cH$ is tangent to $\check{X}$ at b and thus $(c, b) \in V_{\check{X}}$ by the definition of $V_{\check{X}}$.

2.5. The bundle W, see 2.3., contains the modification Y of X along X':

$$Y = p_2^{-1}(G) \qquad \text{and} \qquad f = p_2 | Y\colon Y \to G.$$

If class $X = 0$, i.e. $\dim \check{X} \le N - 2$ the results of 1.6 follow now easily: In this case G does not meet $\check{X}$, i.e. all hyperplanes of the pencil $\{H_t\}_{t \in g}$ intersect X transversally. Hence so does the axis A. Since all points of G are regular values of p_2, $Y = p_2^{-1}(G)$ has n dimensions at every point, in particular there are no singular points. The same reason implies that f has no critical points. It remains to prove that Y is irreducible: Since X is irreducible the open subset $X \backslash X'$ is irreducible; hence so is $Y \backslash Y'$ because this is isomorphic to $X \backslash X'$ under p. The closure of $Y \backslash Y'$ in Y is an irreducible component of Y. The other components of Y (if there are any) must be contained in Y'. Now $\dim_z Y = n$ at every point $z \in Y$, i.e. every component of Y has n dimensions and cannot be contained in Y' which has only $n - 1$ dimensions.

2.6. If class $X > 0$, i.e. if $\check{X} \subset \check{\mathbf{P}}_N$ is a hypersurface the proof of the results of 1.6 is more complicated. The complications are caused by the points $b \in G \cap \check{X} \subset \check{X}_e$. There is exactly one point $c \in X$ such that $(c, b) \in V = V_X = V_{\check{X}}$, because $V'_{\check{X}}$ is mapped isomorphically onto $\check{X}_e$ by π_2, see §2.1. The following tangent spaces are equal:

$$(2.6.1) \qquad T_b({}_cH) = (Tp_2)(T_{(c,b)}W) = (T\pi_2)(T_{(c,b)}V) = T_b\check{X}$$

because of (2.4.1 and 2.4.2).

Proof of (1.6.1). If A did not intersect X transversally, there would be a hyperplane H_b of the pencil $\{H_t\}_{t \in G}$ tangent to X at a point $c \in A$. This means $(c, b) \in V$. On the other hand $c \in A \subset H_b$ dualizes to ${}_cH \supset G \ni b$. Since G intersects $\check{X}$ transversally, so does ${}_cH$, that means $(c, b) \notin V$ by the duality Theorem 2.2.

The projection p_2: $W \to \check{\mathbf{P}}_N$ is transversal to G, i.e. if $(c, b) \in W$ and $b \in G$ the tangent space $T_b\check{\mathbf{P}}_N$ is spanned by $(Tp_2)(T_{(c,b)}W)$ and T_bG.

Proof. If p_2 has maximal rank N at (c, b), $(Tp_2)(T_{(c,b)}W) = T_b\check{\mathbf{P}}_N$ alone suffices. Otherwise, $(c, b) \in V$ (see 2.3.) and therefore $(Tp_2)(T_{(c,b)}W) = T_b\check{X}$ by (2.6.1). The result follows now because G intersects X transversally at b.

Proof of (1.6.2). Since p_2 is transversal to G, the modification $Y = \pi_2^{-1}(G)$ has n dimensions at every point; in particular Y has no singularities. From this it follows that Y is irreducible by exactly the same argument as in the case class $X = 0$: see the last part of 2.5.

Proof of (1.6.3). At every point $(c, b) \in Y$.

$$(2.6.2) \qquad (Tf)(T_{(c,b)}Y) = (Tp_2)(T_{(c,b)}W) \cap T_bG.$$

If $b \in G \backslash \check{X}$, then $(c, b) \notin V$, hence $(Tp_2)(T_{(c,b)}W) = T_b\check{\mathbf{P}}_N$ and (2.6.2) shows that f has maximal rank 1 at (c, b). If $b \in G \cap \check{X}$ the point (c, b) lies in V therefore $(Tp_2)(T_{(c,b)}W) = T_b\check{X}$ by (2.6.1). Since G intersects $\check{X}$ transversally at b, the intersection (2.6.2) is the 0-space, i.e. (c, b) is a critical point of f. At the beginning of this §2.6 it has been remarked that for every $b \in G \cap \check{X}$ there is exactly one $(c, b) \in V$. Therefore no two critical points lie in the same fibre of f.

Proof of (1.6.4). A coordinate description of f in a neighbourhood of a critical point (c, b) will be calculated. Since (c, b) is critical, $(c, b) \in V$ and $b \in G$. The projective coordinates of $\mathbf{P}_N$ are denoted by $x = (x_0 : \ldots : x_N)$, the dual coordinates of $\check{\mathbf{P}}_N$ by $y = (y_0 : \ldots : y_N)$. They are chosen such that $c = (1:0 \ldots :0)$, $b = (0: \ldots :0:1)$ and so that $G \subset \check{\mathbf{P}}_N$ is given by $y_1 = \cdots = y_{N-1} = 0$. The following explicit trivialization of $p_1 \colon W \to X$ over $U = \{x \in X \mid x_0 \neq 0\}$ will be used:

$$U \times \check{\mathbf{P}}_{N-1} \to p_1^{-1}(U), \tag{2.6.3}$$
$$(x, z) \mapsto \left(x, \left(-\sum_{i=1}^{N} x_i z_i : x_0 z_1 : \ldots : x_0 z_N\right)\right)$$

Here $z = (z_1 : \ldots : z_N) \in \mathbf{P}_{N-1}$. Let $(t_1, \ldots, t_n)$ be local holomorphic coordinates of X in a neighbourhood of c. They together with the affine coordinates $\zeta_1 = \frac{z_1}{z_N}, \ldots, \zeta_{N-1} = \frac{z_{N-1}}{z_N}$ of $\check{\mathbf{P}}_{N-1}$ yield the homomorphic coordinates $(t_1, \ldots, t_n, \zeta_1, \ldots, \zeta_{N-1})$ of W in a neighbourhood of (c, b). In a neighbourhood of $b \in \check{\mathbf{P}}_N$ the affine coordinates $\eta_0 = \frac{z_{N-1}}{z_N}$ of $\mathbf{P}_{N-1}$ yield the holomorphic coordinates $(t_1, \ldots, t_n, \zeta_1, \ldots, \zeta_{N-1})$ of W in a
The projection $p_2 \colon W \to \mathbf{P}_N$ has now the following coordinate description:

$$\eta_0 = g(t, \zeta), \quad \eta_1 = \zeta_1, \ldots, \eta_{N-1} = \zeta_{N-1}. \tag{2.6.4}$$

Here $g(t, \zeta)$ is a certain holomorphic function and

$$t \to g(t, 0), \qquad t = (t_1, \ldots, t_n) \tag{2.6.5}$$

is a coordinate description of $f \colon Y \to G$ in a neighbourhood of (c, b). The Jacobian of p_2 (2.6.4) is

$$\begin{bmatrix} \frac{\partial g}{\partial t_1} & \cdots & \frac{\partial g}{\partial t_n} & * \cdots * \\ 0 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & & \ddots \\ 0 & \cdots & 0 & & & 1 \end{bmatrix} \tag{2.6.6}$$

Hence the subset V of W where p_2 fails to have maximal rank is given by

$$\frac{\partial g}{\partial t_1} = \cdots = \frac{\partial g}{\partial t_n} = 0. \tag{2.6.7}$$

Now $\pi_2 = p_2 | V$ has rank $N-1$ at (c, b) (see (2.6.1)). Therefore the Jacobian of the defining eqns (2.6.7) of V together with the Jacobian (2.6.6) of p_2 must have rank $N + n - 1$. This big matrix is

$$\begin{bmatrix} \frac{\partial^2 g}{\partial t_1^2} & \cdots & \frac{\partial^2 g}{\partial t_1 \partial t_n} & * \cdots * \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 g}{\partial t_n \partial t_1} & \cdots & \frac{\partial^2 g}{\partial t_n^2} & * \cdots * \\ 0 & \cdots & 0 & * \cdots * \\ 0 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & & \ddots \\ 0 & \cdots & 0 & & & 1 \end{bmatrix}.$$

It has rank $= N + n - 1$ if and only if the rank of the Hessian matrix of the second derivatives of $t \mapsto g(t, 0)$ has maximal rank n, i.e. if and only if (c, b) is a non-degenerate critical point of f.

§3. THE HOMOLOGY OF HYPERPLANE SECTIONS

3.0. Singular homology with coefficients in an arbitrary principal domain (like $\mathbf{Z}$ or the fields $\mathbf{F}_p$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$) will be used. The following *excision property* will be very convenient:

Let $f\colon (X, A) \to (Y, B)$ be a continuous mapping between pairs of compact Euclidean neighborhood retracts (ENR), such that $f\colon X\backslash A \to Y\backslash B$ is a homeomorphism. Then f induces an isomorphism

$$f_*\colon H_*(X, A) \to H_*(Y, B)$$

of the relative singular homology.

This follows for example from Proposition IV, 8.7 of Dold's book[6]. All spaces which occur in the following are ENR's because they can be embedded in some $\mathbf{R}^s$, they are locally compact and locally contractible, see e.g. [6] IV, 8.12. A Čech type homology theory could also be used. It has the advantage that the excision property stated above holds true for arbitrary pairs of compact Hausdorff spaces.

Many of Lefschetz' intuitive arguments will be made precise by

EHRESMANN'S FIBRATION THEOREM[7]. *Let $f\colon E \to B$ be a proper differentiable mapping between differentiable manifolds E and B without boundary such that $rk\, f = \dim B$ everywhere. Then f fibres E locally trivially over B, i.e. for every point $b \in B$ there is a neighborhood U and a fibre preserving diffeomophism Φ: $f^{-1}(b) \times U \simeq f^{-1}(U)$. If E has a boundary ∂E and in addition $rk(f \mid \partial E) = \dim B$ everywhere f fibres the pair $(E, \partial E)$ locally trivially, i.e. Φ is a fibre preserving diffeomorphism between the pairs $(f^{-1}(b) \times U,\ (f^{-1}(b) \cap \partial E) \times U) \simeq (f^{-1}(U),\ f^{-1}(U) \cap \partial E)$. Similarly if there is a closed submanifold $E' \subset E$ and in addition $rk(f \mid E') = \dim B$ then f fibres the pair (E, E') locally trivially.*

For a proof of the absolute version which can easily be adapted to the relative cases see e.g. [19].

3.1. Let $p\colon Y \to X$ be the modification of X along X', as in §1.2. The homology of Y and X will now be compared. By (1.2.1) and the Künneth theorem there is a canonical isomorphism

$$\text{(3.1.1)} \qquad \begin{aligned} H_q(X') \oplus H_{q-2}(X') &\simeq H_q(X') \otimes H_0(G) \oplus H_{q-2}(X') \otimes H_2(G) \\ &\simeq H_q(X' \times G) = H_q(Y'). \end{aligned}$$

Therefore by restriction to $H_{q-2}(X')$ and composition with the inclusion $Y' \hookrightarrow Y$ there is a canonical homomorphism $\kappa\colon H_{q-2}(X') \to H_q(Y)$.

(3.1.2) *The sequence $0 \to H_{q-2}(X') \xrightarrow{\kappa} H_q(Y) \xrightarrow{p_*} H_q(X) \to 0$ is exact and splits for every q.*

Proof

I. First it is shown that p_* has a right inverse: For a given $x \in H_q(X)$ let $u \in H^{2n-q}(X)$ be its Poincaré-dual, i.e. $x = u \cap [X]$, where $[X] \in H_{2n}(X)$ is the orientation class. Then $p^*(u) \cap [Y] \in H_q(Y)$ and $p_*(p^*(u) \cap [Y]) = u \cap p_*[Y] = u \cap [X] = x$.

II. The exact homology sequences of (Y, Y') and (X, X') are compared:

$$\begin{tikzcd}
H_{q+1}(Y) \arrow[r] \arrow[d, "p_*"] & H_{q+1}(Y, Y') \arrow[r, "\partial_*"] \arrow[d, "p'_*"] & H_q(X') \oplus H_{q-2}(X') \arrow[r] \arrow[d, "pr"] & H_q(Y) \arrow[r] \arrow[d, "p_*"] & H_q(Y, Y') \arrow[d, "p'_*"] \\
H_{q+1}(X) \arrow[r] & H_{q+1}(X, X') \arrow[r, "\partial_*"] & H_q(X') \arrow[r] & H_q(X) \arrow[r] & H_q(X, X').
\end{tikzcd}$$

Here p'_* is an isomorphism because p' is a relative homeomorphism, see (1.2.2) and §3.0. Furthermore $H_q(Y')$ has been replaced by $H_q(X') \oplus H_{q-2}(X')$ using (3.1.1). Diagram chasing (here "p_* is epimorphic" is quite important) yields the desired result.

3.2. Consider now $f: Y \to G$ as in §1.2. Decompose the projective line G (which is a two-sphere) into two closed hemispheres D_+ and D_- such that the critical values of f are contained in the interior $\mathring{D}_+$. Denote

$$G = D_+ \cup D_-, \quad S^1 = D_+ \cap D_-, \quad Y_\pm = f^{-1}(D_\pm), \quad Y_0 = f^{-1}(S^1). \tag{3.2.1}$$

Choose a base point $b \in S^1$.

Through Lefschetz does not state it explicitly the following main lemma is a precise formulation of many of his arguments.

(3.2.2) MAIN LEMMA. *$H_q(Y_+, Y_b) = 0$ if $q \neq n = \dim X = \dim Y$, $H_n(Y_+, Y_b)$ is free of rank $r =$* class *X.*

This lemma will be proved in §5. We shall now show how many of Lefschetz's results follow from this lemma using standard techniques of homology theory, in particular the exact sequences for pairs and triples of spaces.

3.3. To begin with consider the exact homology sequence of the triple $Y \supset Y_+ \supset Y_b$. The homology $H_q(Y, Y_+)$ which occurs in it will be replaced by $H_{q-2}(X_b)$ by means of the following isomorphism

$$H_q(Y, Y_+) \xleftarrow[\cong]{exc.} H_y(Y_-, Y_0) \overset{\Phi_*}{\cong} H_q(X_b \times (D_-, S^1)) \xrightarrow[\cong]{\cdots \times [D_-]} H_{q-2}(X_b). \tag{3.3.1}$$

For the excision isomorphism see §3.0. Since f has no critical values within D_- the Ehresmann fibration theorem (also in §3.0) shows that there is a diffeomorphism

$$\Phi: Y_- \simeq X_b \times D_- \tag{3.3.2}$$

which yields Φ_* in (3.3.1). Finally the canonical orientation of G determines a generator $[D_-]$ of $H_2(D_-, S^1)$. The cross-product with it is an isomorphism because of the Künneth formula. The homology sequence of (Y, Y_+, Y_b) thus becomes the exact sequence

$$\cdots \to H_{q+1}(Y_+, Y_b) \to H_{q+1}(Y, Y_b) \xrightarrow{L} H_{q-1}(X_b) \xrightarrow{\tau} H_q(Y_+, Y_b) \to \cdots \tag{3.3.3}$$

Because of (3.2.2) this sequence decomposes into the isomorphisms

$$L: H_{q+1}(Y, Y_b) \approx H_{q-1}(X_b), \quad q \neq n-1, n \tag{3.3.4}$$

and a 5-term exact sequences containing $H_n(Y_+, Y_b)$.

3.4. The first application of (3.3.4) is a Bertini type theorem:

(3.4.1) *The generic hyperplane section X_b is non-singular and irreducible provided* $\dim X = n \geq 2$.

Proof. Generic means $b \not\in \check{X}$, hence X_b is non-singular because of (1.4.2). Thus "irreducible" is the same as "connected". Since $n \geq 2$ (3.3.4) yields $H_0(Y, Y_b) = H_1(Y, Y_b) = 0$, thus $H_0(Y_b) \approx H_0(Y)$. This implies $X_b \approx Y_b$ is connected because Y is connected according to (1.6.2).

3.5. The second application is to the Euler–Poincaré characteristics e of X, Y, X_b and X'. Using the fact that the alternating sum of the ranks of the modules of a finite exact sequence is zero, (3.1.2) yields

$$e(Y) = e(X) + e(X'), \tag{3.5.1}$$

and (3.3.3) yields $e(Y) - e(Y_b) = e(Y, Y_b) = e(X_b) + (-1)^n r$, hence

$$e(Y) = 2e(X_b) + (-1)^n r \tag{3.5.2}$$

$$e(X) = 2e(X_b) - e(X') + (-1)^n r, \qquad r = \text{class } X, \tag{3.5.3}$$

(compare Lefschetz [L], Chap. III, §11 ($n = 2$) and Chap. V, §9, Théorème XII for arbitrary n). According to Lefschetz, for $n = 2$, this formula is due to J. W. Alexander. For $n = 1$, i.e. for a curve $X \subset \mathbf{P}_n$, the result (3.5.3) is still non-trivial but much older as will now be explained: There is a projection $\mathbf{P}_N \to \mathbf{P}_2$ such that the image of X is a plane curve C which has no singularities but ordinary double points. Let d denote the degree of C and ν the number of double points, let g be the genus of C = *genus of* X. Then by definition $e(X) = 2 - 2g$, furthermore $e(X_b) = d$ because X_b consists of d points, $e(X') = 0$ because X' is empty. Finally, X and C have the same class

$$r = d(d-1) - 2\nu.$$

(This is one of Plücker's formulas, see e.g. Walker's book [17, Chap. IV, 6.2 and Chap. V, 8.2.]) Therefore the result (3.5.3) becomes a well known formula for the genus:

$$g = \frac{(d-1)(d-2)}{2} - \nu \qquad \text{(Clebsch 1864)},$$

see e.g. [17, Chap. VI, Theorem 5.1.].

3.6. The third application yields Lefschetz's famous

THEOREM ON THE HOMOLOGY OF HYPERPLANE SECTIONS.

(3.6.1) $$H_q(X, X_b) = 0 \textit{ for all } q \leq n-1, \qquad n = \dim X,$$

in other words: The inclusion $X_b \hookrightarrow X$ induces isomorphisms of the homology groups in all dimensions strictly less than $n-1$ and an epimorphism of H_{n-1}.

The proof requires a modification of §3.3 which replaces Y_+ and Y_b by their union with Y'. Then (3.3.1) becomes an isomorphism

(3.6.2) $$H_q(Y, Y_+ \cup Y') \simeq H_{q-2}(X_b, X').$$

Furthermore the excision theorem of §3.0 implies that

(3.6.3) $$p_*\colon H(Y, Y_b \cup Y') \simeq H_*(X, X_b)$$

is an isomorphism and finally

(3.6.4) $$H_*(Y_+ \cup Y', Y_b \cup Y') \simeq H_*(Y_+, Y_b)$$

induced by the composed inclusions $(Y_+, Y_b) \hookrightarrow (Y_+, Y_b \cup Y'_+) \hookrightarrow (Y_+ \cup Y', Y_b \cup Y')$. Since $Y_b = X_b \times \{b\}$ is a deformation retract of $Y_b \cup Y'_+ = X_b \times \{b\} \cup X' \times D_+$, the first inclusion induces an isomorphism in the homology, and so does the second one because of the excision property (see §3.0). Thus the homology sequence of $(Y, Y_+ \cup Y', Y_b \cup Y')$ is transformed into the exact sequence

(3.6.5) $$\cdots \to H_{q+2}(Y_+, Y_b) \xrightarrow{p_*} H_{q+2}(X, X_b) \xrightarrow{L'} H_q(X_b, X') \xrightarrow{\tau'} H_{q+1}(Y_+, Y_b) \to \cdots,$$

which replaces (3.3.3). This sequence decomposes into the isomorphisms

(3.6.6) $$L'\colon H_{q+1}(X, X_b) \simeq H_{q-1}(X_b, X'), \qquad q \neq n-1, n$$

and a 5-term exact sequence containing $H_n(Y_+, Y_b)$.

The Lefschetz Theorem (3.6.1) follows now by induction on $n = \dim X$: The beginning $n = 1$ is trivial. Induction from $n-1$ to $n (n \geq 2)$: The hyperplane section X_b is an $(n-1)$-dimensional, irreducible closed subvariety without singularities in $H_b \simeq \mathbf{P}_{N-1}$ (see (3.4.1)), and $X' = X_b \cap A$ is a transversal hyperplane section of X_b. Hence the induction hypothesis applies for (X_b, X'), i.e. $H_q(X_b, X') = 0$ for $q \leq n-2$. The isomorphisms (3.6.6) then yield (3.6.1).

When the universal coefficient theorem is applied to (3.6.1) the corresponding result for the *cohomology* follows:

(3.6.7) $$H^q(X, X_b) = 0 \textit{ for } q \leq n-1, \qquad n = \dim X,$$

in other words: The inclusion X_b induces isomorphisms of the cohomology groups in dimensions strictly less than $n-1$ and a monomorphism of H^{n-1}.

The universal coefficient theorem furthermore shows that the natural epimorphism (R the coefficient ring)

$$H^n(X, X_b; R) \simeq Hom(H_n(X, X_b), R) \tag{3.6.8}$$

is an isomorphism, and hence $H^n(X, X_b; \mathbf{Z})$ is free. By the Poincaré–Lefschetz duality theorem these results are equivalent to

$$H_q(X\backslash X_b) = 0 \textit{ for } q \geq n+1 \textit{ and } H_n(X\backslash X_b, \mathbf{Z}) \textit{ is free.} \tag{3.6.9}$$

This proof of (3.6.1) is essentially Lefschetz's original proof as in [L] Chap. V, §3. Lefschetz's proof is difficult to understand because he did not use exact sequences. He constructed L (3.6.6) or rather L^{-1} quite explicitly for chains. He calls $L^{-1}(x)$ the "locus of x as b varies": see [L] Chap. II, §11 ($n = 2$) and Chap. V, §3–5 (n arbitrary).

3.7. Using 1.3 the results about *hyperplane* sections can be generalized to *hypersurface* sections. To be more precise:

(3.7.1) *Let $X \subset \mathbf{P}_N$ be a smooth irreducible n-dimensional variety, let $F \subset \mathbf{P}_N$ be a hypersurface such that all points of $F \cap X$ are simple points of F and F intersects X transversally. Then $H_q(X, X \cap F) = 0$ for $q \leq n-1$, i.e., the inclusion $X \cap F \hookrightarrow X$ induces isomorphisms of all homology groups in dimensions $\leq n-2$ and an epimorphism in dimension $= n-1$.*

Using (3.7.1) the topology of complete intersections can be compared with the topology of projective spaces: A subset $Y \subset \mathbf{P}_N$ is called a *smooth complete intersection*, if $Y = F_1 \cap \cdots \cap F_r$ is the intersection of hypersurfaces $F_1, \ldots, F_r \subset \mathbf{P}_N$ such that F_1 is smooth, the points of $F_1 \cap F_2$ are simple points of F_2 and F_2 intersects F_1 transversally, the points of $F_1 \cap F_2 \cap F_3$ are simple points of F_3 and F_3 intersects $F_1 \cap F_2$ transversally and so on. In this case Y is a smooth $(N-r)$-dimensional variety. Apply now (3.7.1) first to $X = \mathbf{P}_N$ and $F = F_1$ then to $X = F_1$ and $F = F_2$, then to $X = F_1 \cap F_2$ and $F = F_3$ and so on:

(3.7.2) *If $Y \subset \mathbf{P}_N$ is an n-dimensional smooth complete intersection, then $H_q(\mathbf{P}_N, Y) = 0$ for $q \leq n$, i.e. $Y \to \mathbf{P}_N$ induces isomorphism of all homology groups in dimensions $\leq n-1$ and an epimorphism in dimension $= n$.*

This imposes strong topological restrictions on n-dimensional varieties Y which can be embedded as smooth complete intersections: Except in the middle dimension n the homology groups of Y and $\mathbf{P}_N$ are isomorphic (for dimensions $> n$ this follows by Poincaré duality). Furthermore if n is even the nth Betti number of Y is ≥ 1. If, e.g. C is a smooth curve of genus > 0 the product $C \times \mathbf{P}_n$, $n \geq 1$, is not a smooth complete intersection because its first Betti number is > 0, the products $\mathbf{P}_q \times \mathbf{P}_r$ except for $\mathbf{P}_1 \times \mathbf{P}_1$ are not smooth complete intersections because their second Betti number is 2 and not 1.

3.8. Consider the connecting homomorphism $\partial_*: H_n(Y_+, Y_b) \to H_{n-1}(Y_b) \overset{p}{\simeq} H_{n-1}(X_b)$. Its image

$$V = \partial_*(H_n(Y_+, Y_b))$$

is called the module of "vanishing cycles". The exact homology sequences of (Y_+, Y_b) and (X, X_b) form the following commutative diagram

$$
\begin{tikzcd}
H_n(Y_+, Y_b) \arrow[r, "\partial_*"] \arrow[d, "p_1"] & H_{n-1}(Y_b) \arrow[r] \arrow[d, "\simeq" left, "p_2"] & H_{n-1}(Y_+) \arrow[r] \arrow[d, "\simeq" left, "p_3"] & 0 \\
H_n(X, X_b) \arrow[r, "\partial_*"] & H_{n-1}(X_b) \arrow[r, "i_*"] & H_{n-1}(X) \arrow[r] & 0.
\end{tikzcd}
\tag{3.8.1}
$$

All vertical homomorphisms are induced by restrictions of $p\colon Y \to X$. The left hand one p_1 is epimorphic because it occurs in the exact sequence (3.6.5) and the following term $H_{n-2}(X_b, X') = 0$ according to (3.6.1). The middle one p_2 is an isomorphism. Hence the Five Lemma implies that p_3 is also an isomorphism. This diagram shows that

$$
\begin{aligned}
V &= \text{image } (\partial_*\colon H_n(X, X_b) \to H_{n-1}(X_b)) \\
&= \text{kernel } (i_*\colon H_{n-1}(X_b) \to H_{n-1}(X)),
\end{aligned}
\tag{3.8.2}
$$

and

$$
rk\, H_{n-1}(X_b) = rk\, V + rk\, H_{n-1}(X). \tag{3.8.3}
$$

3.9. When §3.8 is translated into cohomology we get the commutative diagram with exact lines

$$
\begin{tikzcd}
H^n(Y_+, Y_b) & H^{n-1}(Y_b) \arrow[l, "\delta_*"'] & H^{n-1}(Y_+) \arrow[l] & 0 \arrow[l] \\
H^n(X, X_b) \arrow[u, "mono"'] & H^{n-1}(X_b) \arrow[l, "\delta_*"'] \arrow[u, "\simeq"'] & H^{n-1}(X) \arrow[l, "i_*"'] \arrow[u] & 0. \arrow[l]
\end{tikzcd}
$$

This diagram shows (if X_b and Y_b are identified as usual)

$$
\begin{aligned}
I^*\colon &= \text{kernel } (\delta^*\colon H^{n-1}(Y_b) \to H^n(Y_+, Y_b)) \\
&= \text{kernel } (\delta^*\colon H^{n-1}(X_b) \to H^n(X, X_b)) \\
&= \text{image } (i^*\colon H^{n-1}(X) \to H^{n-1}(X_b)).
\end{aligned}
\tag{3.9.1}
$$

I^* is called the modulé of "invariant cocycles". The module I of invariant cycles is defined to be the Poincaré dual of I^*, i.e.

$$
I\colon = \{u \cap [X_b] \mid u \in I^*\} \subset H_{n-1}(X_b). \tag{3.9.2}
$$

The last description of I^* yields by Poincaré duality

$$
I = \text{image } (i_!\colon H_{n+1}(X) \to H_{n-1}(X_b)). \tag{3.9.3}
$$

Here $i_!$ denotes the Umkehr homomorphism (transfer), i.e. the Poincaré dual of i^*, see e.g. Dold[6, Chap. VIII, 10]. Since i^* is injective, $i_!$ is also injective, in particular

(3.9.4) $$\operatorname{rank} I = \operatorname{rank} H_{n+1}(X) = \operatorname{rank} H_{n-1}(X).$$

The last equality comes from Poincaré duality. The first description of I^* (3.9.1) together with $H^n(Y_+, Y_b) \simeq \operatorname{Hom}(H_n(Y_+, Y_b), R)$ (here R denotes the coefficient ring and the isomorphism comes from the universal coefficient theorem because $H_{n-1}(Y_+, Y_b) = 0$ according to 3.2.2) yields $I^* = \{u \in H^{n-1}(Y_b) \mid \langle u, x\rangle = 0$ *for every* $x \in V\}$. Here $\langle -, -\rangle$ denotes the Kronecker pairing between cohomology and homology. By Poincaré duality the Kronecker pairing becomes the intersection form

$$H_{n-1}(X_b) \times H_{n-1}(X_b) \to R,$$

which will also be denoted by $\langle -, -\rangle$; thus:

(3.9.5) $$I = \{y \in H_{n-1}(X_b) \mid \langle y, x\rangle = 0 \quad \text{for every } x \in V\}.$$

If coefficients in a field are taken, the intersection form is non-degenerate by Poincaré duality. Hence (3.9.5) implies

(3.9.6) $$\operatorname{rank} I + \operatorname{rank} V = \operatorname{rank} H_{n-1}(X_b).$$

The rank formulas (3.8.3), (3.9.4) and (3.9.6) can be found in [L] Chap. III, §3 ($n = 2$) and Chap. V, §6 (n arbitrary).

§4. THE HARD LEFSCHETZ THEOREM

Lefschetz derives the rank formulas (3.8.3), (3.9.4) and especially (3.9.6) from a much stronger result namely: $H_{n-1}(X_b)$ *is the direct sum of* I *and* V. (This would follow from (3.9.5) if the intersection form were definite.) This stronger result is nowadays called the "Hard Lefschetz Theorem". In this chapter several equivalent formulations of this theorem and consequences of it will be discussed. A proof will not be given.

4.1. Let $u \in H^2(X)$ denote the Poincaré dual of the fundamental class $[X_b] \in H_{2n-2}(X)$ of the hyperplane section X_b, i.e.

$$u \cap [X] = [X_b].$$

The homological expression for the intersection with X_b is the cap-product with u. It factors through X_b, i.e.

(4.1.1) $$u \cap \cdots : H_q(X) \xrightarrow{i_!} H_{q-2}(X_b) \xrightarrow{i_*} H_q(X).$$

THEOREM. *If field coefficients are chosen, the following statements are equivalent*:

(4.1.2) $V \cap I = 0$

(4.1.3) $V \oplus I = H_{n-1}(X_b)$

(4.1.4) i_*: $H_{n-1}(X_b) \to H_{n-1}(X)$ *maps* I *isomorphically onto* $H_{n-1}(X)$.

(4.1.5) $H_{n+1}(X) \simeq H_{n-1}(X)$, $\quad x \mapsto u \cap x$, $\quad$ *is an isomorphism.*

(4.1.6) *The restriction of the intersection form* $\langle -, - \rangle$ *from* $H_{n-1}(X_b)$ *to* V *remains non-degenerate.*

(4.1.7) *The restriction of* $\langle -, - \rangle$ *to* I *remains non-degenerate.*

Proof of the equivalences: (4.1.2) and (4.1.3) are equivalent because of (3.9.6). Since $i_*: H_{n-1}(X_b) \to H_{n-1}(X)$ is epimorphic (3.8.1) and maps V to 0 (3.8.2), the statement (4.1.4) follows from (4.1.3). According to §3.9 $i_!$ is monomorphic and image $i_! = I$; thus $u \cap \cdots = i_* \circ i_!$ is monomorphic because of (4.1.4). Then (4.1.5) follows because $H_{n+1}(X)$ and $H_{n-1}(X)$ are isomorphic by Poincaré duality. Vice versa: If (4.1.5) holds true, $i_*(I) = H_{n-1}(X)$, therefore $i_* | I$ is an isomorphism because of (3.9.4). Hence (4.1.4) follows from (4.1.5). Since $i_*(V) = 0$ (4.1.4) implies (4.1.2). Thus (4.1.2–5) are equivalent.

(4.1.3) and (3.9.5) imply that the intersection form $\langle -, - \rangle$ on $H_{n-1}(X_b)$ splits into the direct sum of its restrictions to V and I,

$$\langle -, - \rangle = \langle -, - \rangle_V \oplus \langle -, - \rangle_I.$$

Since $\langle -, - \rangle$ is non-degenerate by Poincaré duality, the direct summands must also be non-degenerate. Thus (4.1.6) and (4.1.7) follow from (4.1.3). Vice versa (4.1.6) or (4.1.7) implies (4.1.2): Assume $z \in V \cap I$. Then $\langle z, v \rangle = \langle z, v \rangle_V = 0$ for every $v \in V$ and $\langle c, z \rangle = \langle c, z \rangle_I = 0$ for every $c \in I$ according to (3.9.5). The first statement together with (4.1.6) or the second statement together with (4.1.7) both imply $z = 0$, i.e. $V \cap I = 0$.

(4.1.8) THE HARD LEFSCHETZ THEOREM. *The statements* (4.1.2)–(4.1.7) *are true if coefficients in a field of characteristic zero are chosen.*

Lefschetz claims that (4.1.2) and (4.1.3) hold true for integer coefficients, see [L] Chap. II, §13 and 18 for $n = 2$ and Chap. V, §7 for arbitrary n. But his proof is difficult to understand and seems to be incomplete even for field coefficients. At present I don't know a complete topological proof. The only complete proof comes from Hodge's theory of harmonic integrals (forms), see §4.6 below, where the *co*homological version is presented.

The other statements (4.1.4), (4.1.6) and (4.1.7) are also due to Lefschetz [L], Chap. II, §19 and Chap. II, §3 and §5. For (4.1.5) Lefschetz has a better version: see (4.3.2) below.

For the rest of this §4 *coefficients in a field of characteristic zero are chosen so that the statements* (4.1.2)–(4.1.7) *hold true.*

4.2. Iterate the sequence $X \supset X_b \supset X'$ to

$$X = X_0 \supset X_b = X_1 \supset X_2 = X' \supset X_3 \cdots \supset X_n \supset X_{n+1} = 0 \tag{4.2.1}$$

so that X_q is a generic hyperplane section of X_{q-1}, hence

$$\dim X_q = n - q.$$

Denote the inclusions by

$$i_q: X_q \hookrightarrow X.$$

Define the submodule

$$I(X_q) \subset H_{n-q}(X_q)$$

of *invariant "cycles"* for the pair $X_q \subset X_{q-1}$ in the same way as for the pair $X_b \subset X$, see §3.9. Then (3.9.3), (4.1.4) and (4.1.7) can be generalized to

(4.2.2) $(i_q)_!\colon H_{n+q}(X) \to H_{n-q}(X_q)$ *maps* $H_{n+q}(X)$ *isomorphically onto* $I(X_q)$.

(4.2.3) $(i_q)_*\colon H_{n-q}(X_q) \to H_{n-q}(X)$ *maps* $I(X_q)$ *isomorphically onto* $H_{n-q}(X)$.

(4.2.4) *The restriction of the intersection form* $\langle -, - \rangle$ *from* $H_{n-q}(X_q)$ *to* $I(X_q)$ *remains non-degenerate.*

The isomorphism $(i_q)_*\colon I(X_q) \to H_{n-q}(X)$ carries this form to a non-degenerate bilinear form Q on $H_{n-q}(X)$. The form Q is symmetric if $n-q$ is even and skew-symmetric if $n-q$ is odd. Since non-degenerate skew symmetric forms can only exist on even-dimensional vector spaces the following consequence is obtained:

(4.2.5) *The odd-dimensional Betti numbers of* X *are even.*

This result and its proof are essentially due to Lefschetz, see [L] Chap. II, §19. As Lefschetz already points out this result shows: In contrast to real surfaces, for $l > 1$ not every closed oriented 2l-dimensional real manifold is homeomorphic to a complex projective manifold. Even certain compact *complex* manifolds like the Hopf manifolds (see [3, p. 3]) which are homeomorphic to $S^{2m-1} \times S^1$ are excluded this way.

4.3. The q-th power $u^q \in H^{2q}(X)$ is Poincaré dual to the fundamental class $[X_q] \in H_{2n-2q}(X)$ of X_q. Therefore the decomposition (4.1.1) generalizes to

(4.3.1) $$u^q \cap \cdots : H_k(X) \xrightarrow{(i_q)_!} H_{k-2q}(X_q) \xrightarrow{(i_q)_*} H_{k-2q}(X),$$

and (4.2.2) and (4.2.3) imply the following generalization of (4.1.5):

(4.3.2) *For every* $q = 1, \ldots, n$ *the cap-product with the* qth *power* u^q *is an isomorphism*

$$H_{n+q}(X) \xrightarrow{\cong} H_{n-q}(X), \qquad x \mapsto u^q \cap x.$$

This version of the Hard Lefschetz Theorem and its proof are essentially due to Lefschetz himself [L] Chap. V, §8, Théorème VII and VIII. The following reformulation is due to Hodge[9, Chap. IV, No. 44].

4.4. An element $x \in H_{n+q}(X)$, $0 \le q \le n$, is called *primitive* if $u^{q+1} \cap x = 0$. ($u^q \cap x = 0$ would imply $x = 0$ by (4.3.2).) The result (4.3.2) and hence the Hard Lefschetz Theorem is equivalent to the following

Primitive Decomposition. *Every element* $x \in H_{n+q}(X)$ *can be written uniquely as*

(4.4.1a) $$x = x_0 + u \cap x_1 + u^2 \cap x_2 + \cdots$$

and every element $x \in H_{n-q}(X)$ *as*

$$x = u^q \cap x_0 + u^{q+1} \cap x_1 + u^{q+2} \cap x_2 + \cdots \tag{4.4.1b}$$

where the $x_i \in H_{n+q+2i}(X)$ *are primitive, and* $q \geq 0$.

Proof. The cap-product with u^q obviously transforms (4.4.1a) into (4.4.1b). Since the representations are unique, $u^q \cap \cdots$ is an isomorphism and thus (4.4.1a) and (4.4.1b) implies (4.3.2). Vice versa (4.4.1a) follows from (4.3.2) by induction beginning with $q = n$ and $q = n-1$ where every element is primitive. For the induction step from $n+q+2$ to $n+q$ it suffices to show that every $x \in H_{n+q}(X)$ can be written uniquely as

$$x = x_0 + u \cap y \quad \text{with } x_0 \text{ primitive,} \tag{4.4.2}$$

because the induction hypothesis applied to y then yields the decomposition (4.4.1a). In order to prove (4.4.2) consider $u^{q+1} \cap x$. According to (4.3.2) there is exactly one $y \in H_{n+q+2}(X)$ with $u^{q+2} \cap y = u^{q+1} \cap x$, and thus $x_0 = x - u \cap y$ is primitive. In order to show the uniqueness assume $0 = x_0 + u \cap y$ with x_0 primitive. Then $u^{q+1} \cap x_0 = 0$, hence $u^{q+2} \cap y = 0$, and (4.3.2) implies $y = 0$, hence $x_0 = 0$. The isomorphism $u^q \cap \cdots$ (4.3.2) applied to the unique decomposition (4.4.1a) yields the unique decomposition (4.4.1b).

The primitive decomposition shows that the homology of X is completely determined by the submodules $P_{n+q}(X) \subset H_{n+q}(X)$, $0 \leq q \leq n$, of the primitive elements. The intermediate result (4.4.2) implies

$$\dim P_{n+q} = b_{n+q} - b_{n+q+2} = b_{n-q} - b_{n-q-2} \tag{4.4.3}$$

($b_i = i$-th Betti number of X). Since $\dim P_{n+q} \geq 0$, the Betti numbers form two increasing sequences

$$\begin{aligned} &1 = b_0 \leq b_2 \leq \cdots \leq b_{2i}, && \text{for every } i \text{ with } 2i \leq n \\ &b_1 \leq b_3 \leq \cdots \leq b_{2i+1}, && \text{for every } i \text{ with } 2i+1 \leq n \end{aligned} \tag{4.4.4}$$

Like (4.2.5) this obviously restricts the topological possibilities for projective manifolds.

Remark. Our (4.4.1) is not exactly Hodge's formulation because he uses the "effective cycles" $y \in H_{n-q}(X)$, defined by $u \cap y = 0$, rather than the primitive elements $x \in H_{n+q}(X)$. Since $u^q \cap \cdots$ is an isomorphism (4.3.2) and x is primitive if and only if $u^q \cap x$ is effective it is easy to translate (4.4.1) into a formulation using "effective cycles". The term *primitive* is due to Weil[17].

4.5. The Lie algebra sl_2 of all (2×2)-matrices with trace zero is three dimensional and has a basis consisting of

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Its commutor relations are ($[xy] = xy - yx$):

$$[eh] = -2e, \quad [fh] = 2f, \quad [ef] = h. \tag{4.5.1}$$

Its representations are well known, see e.g. Jacobson[10, Chap. III, §8].

Consider now the endomorphisms of $H_*(X)$

$$\begin{aligned} &f\colon H_j(X) \to H_{j-2}(X), && fx = u \cap x \\ &h\colon H_j(X) \to H_j(X), && hx = (j-n)x. \end{aligned} \tag{4.5.2}$$

Obviously $[fh] = 2f$. The primitive decomposition and hence the Hard Lefschetz Theorem is equivalent to:

(4.5.3) *There is an homomorphism* $e\colon H_j(X) \to H_{j+2}(X)$ *which together with* f, h (4.5.2) *satisfies all commutator relations* (4.5.1), *in other words*: $H_*(X)$ *is an* sl_2-*module.*

Proof. Using the primitive decomposition (4.4.1) it suffices to consider elements of the form

$$u^r \cap x, \quad x \in P_{n+q}(X),$$

in order to define e and to check (4.5.1). The definition is

$$e(u^r \cap x) = r(q - r + 1)u^{r-1} \cap x, \tag{4.5.4}$$

and the checking is easy. Vice versa the representation theory of sl_2 implies the primitive decomposition (4.4.1) in the following way: Like any sl_2-module $H_*(X)$ is a direct sum of irreducible sl_2-modules $A_1 \oplus \cdots \oplus A_s$. Up to isomorphism an irreducible sl_2-module is determined by its dimension: Let $\dim A_i = d_i + 1$. Then there is an element $x_i \in A_i$ so that $\{x_i, fx_i, \ldots, f^{d_i}x_i\}$ is a base of A_i, $f^{d+1}x_i = 0$ and $hx_i = d_i x_i$. The definition (4.5.2) implies that $f^r x_i \in H_{n+d_i-2r}(X)$. Thus $H_p(X) = \bigoplus_{i=1}^{s} H_p(X) \cap A_i$ has the basis $\{f^{q_i}x_i \mid d_i - 2q_i = p\}$. This yields the primitive decomposition.

Remark. The $(d+1)$-dimensional irreducible sl_2-module occurs $\dim P_{n+d}(X)$ times as direct summand in $H_*(X)$. Therefore (4.4.3) may be interpreted in the following way: The Betti numbers of X and the structure of $H_*(X)$ as sl_2-module (up to isomorphism) determine one another.

4.6. The three versions (4.3.2), (4.4.1) and (4.5.3) of the Hard Lefschetz Theorem are easily translated into *co*homology by the Poincaré duality theorem. Then they run as follows:

(4.6.1) ~ (4.3.2) *For every* $q = 1, \ldots, n$ *the cup product with the* q-th *power of* $u \in H^2(X)$ *is an isomorphism*

$$H^{n-q}(X) \xrightarrow{\cong} H^{n+q}(X), \qquad x \mapsto u^q \cup x.$$

A cohomology class $x \in H^{n-q}(X)$ is called primitive if the Poincaré dual homology class $x \cap [X] \in H^{n+q}(X)$ is primitive, i.e. if $u^{q+1} \cup x = 0$.

(4.6.2) ~ (4.4.1) Primitive decomposition: *Every element* $x \in H^{n-q}(X)$ *can be written uniquely as*

$$x = x_0 + u \cup x_1 + u^2 \cup x_2 + \cdots,$$

and every element $x \in H^{n+q}(X)$ *as*

$$x = u^q \cup x_0 + u^{q+1} \cup x_1 + u^{q+2} \cup x_2 + \cdots,$$

where the $x_i \in H^{n-q-2i}(X)$ *are primitive.*

The Poincaré duals of the endomorphisms e, f, h: $H_*(X) \to H_*(X)$, see §4.5, are

(4.6.3)

$$\begin{array}{lll} \Lambda; H^j(X) \to H^{j-2}(X), & u^r \cup x \mapsto r(q-r+1)u^{r-1} \cup x, & x \in H^{n-q}(X) \text{ primitive} \\ L\colon H^j(X) \to H^{j+2}(X), & x \mapsto u \cup x & \\ H\colon H^j(X) \to H^j(X), & x \mapsto (n-j)x. & \end{array}$$

(4.6.4) ~ (4.5.1) and (4.5.3): $[\Lambda H] = 2\Lambda$, $[LH] = -2L$, $[\Lambda L] = H$, i.e. $H^*(X)$ is an sl_2-module.

Hodge proves (4.6.2) for the coefficient field **C** using his theory of harmonic integrals: see [9, Chap. IV, §42–44]. For a more modern presentation which explicitly includes (4.6.1), (4.6.2), the operators (4.6.3) and their commutators (4.6.4) see Weil[17, Chap. IV, Nos. 6 and 8]. Chern[4] seems to be the first who saw the representation theoretical aspect of this theory. See also Cornalba–Griffith[5] for a recent survey of transcendental methods.

§5. THE TOPOLOGY OF HOLOMORPHIC FUNCTIONS WITH NON-DEGENERATE CRITICAL POINTS

This chapter deals with the holomorphic analog of the (finite dimensional) Morse theory. Actually the holomorphic case is older than the real Morse theory because all ideas occur already in [L].

5.1. Let f: $Y \to G$ be a holomorphic mapping between an n-dimensional compact complex manifold Y and a projective line G, such that all critical points $x_1, \ldots, x_r$ of f are non-degenerate and no two lie in the same fibre, compare 1.2 and 1.6. Decompose G into the closed upper and lower hemispheres D_+ and D_- so that all critical values $t_1, \ldots, t_r$ of f are interior points of D_+. A regular value $b \in \partial D_+$ serves as base point. Let

$$Y_+ = f^{-1}(D_+) \quad \text{and} \quad Y_b = f^{-1}(b).$$

In this situation the Main Lemma of §3.2 holds true:

(5.1.1) $$H_q(Y_+, Y_b) = 0 \textit{ if } q \neq n$$

(5.1.2) $$H_n(Y_+, Y_b) \textit{ is free of rank } r.$$

The following proof of (5.1.1) and (5.1.2) will also be the starting point for the investigation of invariant and vanishing cycles in the following chapters.

5.2. By choosing a suitable holomorphic coordinate t the hemisphere D_+ is identified with the closed unit disk in C so that b corresponds to 1. Small disks D_i with center t_i, $i = 1, \dots, r$, and radius ρ are chosen so that they are mutually disjoint and contained in D_+, see Fig. 1. The investigation of (Y_+, Y_b) is carried through in three steps: First (Y_+, Y_b) is reduced by a localization in the base to (T_i, F_i) where

(5.2.1) $$T_i = f^{-1}(D_i) \quad \text{and} \quad F_i = f^{-1}(t_i + \rho).$$

Then one localizes in the total space: Since x_i is a non-degenerate critical point of f in a neighbourhood B of x_i local holomorphic coordinates $(z_1, \dots, z_n)$ of Y can be chosen so that $f|B$ has the coordinate description

(5.2.2) $$f(z) = t_i + z_1^2 + \cdots + z_n^2.$$

The pair (T_i, F_i) is reduced to (T, F) where

(5.2.3) $$T = T_i \cap B \quad \text{and} \quad F = F_i \cap B,$$

see Fig. 2 below. Finally the homology and homotopy of (T, F) is computed using the explicit coordinate description (5.2.2).

5.3. In D_+ C^∞-embedded intervals l_i from b to $t_i + \rho$ are chosen so that $l = \bigcup_{i=1}^{r} l_i$ can be contracted within itself to $\{b\}$ and D_+ can be contracted to $k = l \cup \bigcup_{i=1}^{r} D_i$, see the following figure ($r = 3$):

(5.3.1) *The fibre Y_b is a strong deformation retract of $L = f^{-1}(l)$ and $K = f^{-1}(k)$ is a strong deformation retract of Y_+, hence the inclusions*

$$(Y_+, Y_b) \longrightarrow (Y_+, L) \longleftarrow (K, L)$$

induce isomorphisms of all homology and homotopy groups.

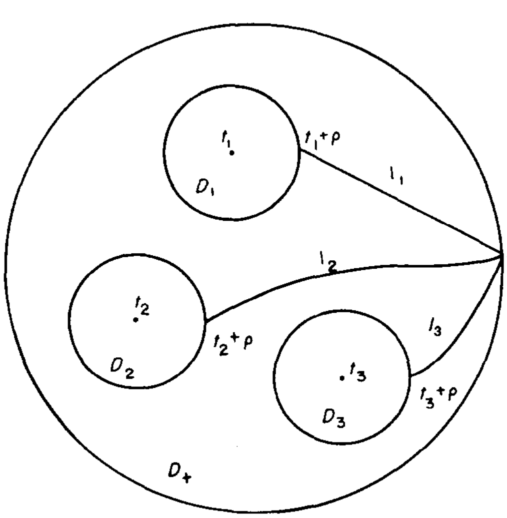

Fig. 1

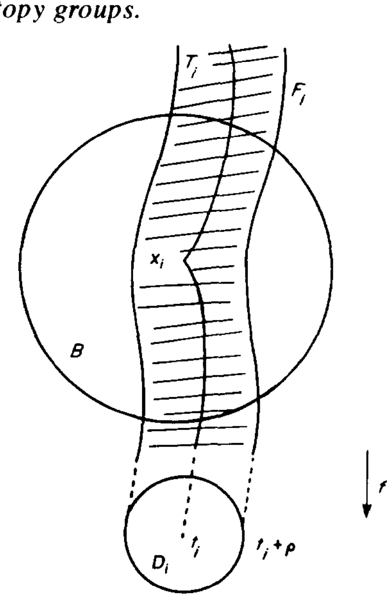

Fig. 2

Proof. According to Ehresmann's fibration theorem $f: Y_+\backslash f^{-1}\{t_1,\dots,t_r\}\to D_+\backslash\{t_1,\dots,t_r\}$ is a C^∞ locally trivial fibre bundle. Since $l\subset D_+\backslash\{t_1,\dots,t_r\}$ $f: L\to l$ is a subbundle. The homotopy covering theorem, see e.g. Steenrod[14, §11.3], implies: The contraction from l to $\{b\}$ can be lifted so that Y_b becomes a strong deformation retract of L. Similarly the contraction of $D_+\backslash\{t_1,\dots,t_r\}$ to $l\cup\bigcup_{i=1}^{r}(D_i\backslash t_i)$ can be lifted so that $L\cup\bigcup_{i=1}^{r}(T_i\backslash f^{-1}(t_i))$ becomes a strong deformation retract of $Y_+\backslash f^{-1}\{t_1,\dots,t_r\}$. Since the t_i are interior points of k the singular fibres can be filled in so that K is a strong deformation retract of Y_+.

In order to reduce the investigation from (Y_+, Y_b) to (T_i, F_i), see (5.2.1), observe that the inclusion $(\bigcup_{i=1}^{r} T_i, \bigcup_{i=1}^{r} F_i)\to(K, L)$ is an excision, i.e. induces an isomorphism in homology. Since the union is disjoint, (5.3.1) finally yields:

(5.3.2) *The inclusions induce isomorphisms*

$$\bigoplus_{i=1}^{r} H_*(T_i, F_i)\xrightarrow{\cong} H_*(Y_+, L)\xleftarrow{\cong} H_*(Y_+, Y_b).$$

5.4. There is exactly one critical point x_i of f within T_i. In a neighbourhood of x_i holomorphic coordinates $(z_1,\dots,z_n)$ of Y are chosen so that $x_i=(0,\dots,0)$ and f is described by (5.2.2). If $\epsilon>0$ is small enough the ball

$$B=\left\{z\in\mathbf{C}^n \,\middle|\, \|z\|^2=|z_1|^2+\cdots+|z_n|^2\le\epsilon^2\right\}$$

is contained in the range of the coordinates. In the following the corresponding subset of Y will also be denoted by B. The radius ρ of D_i must be shrunk so that $\rho<\epsilon^2$. The result of the second localization step from (T_i, F_i) to (T, F), see (5.2.3), is

(5.4.1) *The inclusion $(T, F)\to(T_i, F_i)$ induces isomorphisms for the homology.*

Proof. Let $B=\{z\in B \mid \|z\|=\epsilon\}$, $T'=T\cap\partial B$ and $F'=F\cap\partial B$. Consider the diagram of inclusions

$$\begin{tikzcd}
(T, F) \arrow[r] \arrow[d] & (T_i, F_i) \arrow[d] \\
(T, T'\cup F) \arrow[r] & (T_i, \overline{T_i\backslash B}\cup F_i).
\end{tikzcd}$$

The bottom inclusion is an excision. The following result (5.4.2) implies that both vertical inclusions also induce isomorphisms for the homology. Hence (5.4.1) follows.

(5.4.2) *$F_i\backslash\mathring{B}$ is a strong deformation retract of $T_i\backslash\mathring{B}$ and F' is a strong deformation retract of T'.*

The real analytic mapping f has maximal rank $=2$ everywhere on $T_i\backslash\mathring{B}$ and its restriction $f|\partial B$ has also maximal rank $=2$ on the (partial) boundary $T_i\cap\partial B=T'$. Hence Ehresmann's fibration theorem for manifolds with boundary yields a fibre preserving diffeomorphism between the pairs $(T_i\backslash\mathring{B}, \partial T)$ and $(F_i\backslash\mathring{B}, \partial F)\times D_i$. Since D_i can be contracted onto $t_i+\rho$, this implies (5.4.2).

5.5. For the final step the following explicit coordinate descriptions will be used (This description is due to J. Leray. It has first been published by Fáry[8], §6.):

(5.5.1) $$T=\{z\in\mathbf{C}^n \mid |z_1|^2+\cdots+|z_n|^2\le\epsilon^2 \textit{ and } |z_1{}^2+\cdots+z_n{}^2|\le\rho\}$$

(5.5.2) $$F=\{z\in T \mid z_1{}^2+\cdots+z_n{}^2=\rho\}$$

(5.5.3) $$f(z)=t_i+z_1{}^2+\cdots+z_n{}^2.$$

(5.5.1) shows that T can be linearly contracted onto the origin. Therefore the connecting homomorphism

(5.5.4) $$\partial_*\colon H_q(T,F)\xrightarrow{\ \simeq\ } H_{q-1}(F)\quad \text{for } q\neq 0$$

is an isomorphism and $H_0(T,F)=0$.

There is a well known real analytic diffeomomorphism between F and the space

(5.5.5) $$Q=\{(u,v)\in\mathbf{R}^n\times\mathbf{R}^n \mid \|u\|=1,\quad \|v\|\le 1,\quad \langle u,v\rangle=0\}$$

of all tangent vectors of length ≤ 1 of the unit sphere S^{n-1} in $\mathbf{R}^n$. Here $\langle u,v\rangle=\sum_{\nu=1}^{n}u_\nu v_\nu$ denotes the usual Euclidean inner product and $\|u\|=\sqrt{\langle u,u\rangle}$ the corresponding norm. This diffeomorphism is given in the following way: Decompose $z_\nu=x_\nu+iy_\nu$ into its real and imaginary part. Let $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)\in\mathbf{R}^n$. Then from (5.5.1) and (5.5.2) $F=\{(x,y)\in\mathbf{R}^n\times\mathbf{R}^n \mid \|x\|^2+\|y\|^2\le\epsilon^2,\ \|x\|^2-\|y\|^2=\rho,\ \langle x,y\rangle=0\}$. This implies $\|y\|\le\sqrt{\left(\frac{\epsilon^2-\rho}{2}\right)}=:\sigma$. Then $F\simeq Q$ is given by

(5.5.6) $$u+iv=\frac{x}{\|x\|}+\frac{i}{\sigma}y,\quad \text{inverse: } x+iy=\sqrt{(\sigma^2\|v\|^2+\rho)}\cdot u+i\sigma v.$$

This diffeomorphism maps the real part of F, namely the sphere

(5.5.7) $$S^{n-1}=\{z\in F \mid \textit{all } z_\nu \textit{ real}\}$$

onto the zero section $Q_0=\{(u,0)\in Q\}$ of Q. Therefore S^{n-1} is a strong deformation retract of F and the homology of F is

(5.5.8) *$H_{q-1}(F)=0$ for $q\neq 1$, $\neq n$, $H_0(F)$ and $H_{n-1}(F)$ are free of rank* 1, *an orientation of S^{n-1} determines a generator of $H_{n-1}(F)$.*

Using (5.5.4) this is translated into the relative homology:

(5.5.9) *$H_q(T,F)=0$ for $q\neq n$, $H_n(T,F)$ is free of rank* 1 *and an orientation of the real n-disk*

$$\Delta=\{z\in T \mid \textit{all } z_\nu \textit{ real}\}$$

represents a generator of $H_n(T,F)$.

The results (5.1.1) and (5.1.2) are now easily deduced from (5.5.9), (5.4.1) and (5.3.2).

Remark. Later, for the proof of (6.5.2) an explicit retraction $R\colon T' \to F'(\hookrightarrow F)$, see (4.4.2) will be needed: The coordinate description

$T' = \{z \in \mathbf{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 = \epsilon^2 \text{ and } |z_1^2 + \cdots + z_n^2| \leq \rho\}$

$F' = \{z \in T' \mid z_1^2 + \cdots + z_n^2 = \rho\}$

is used. Let $z \in T'$ be given. Represent $f(z) = t_i + r \cdot e^{2\pi i\varphi}$ in polar coordinates, define $z' = e^{-\pi i\varphi}z$ (so that $f(z') = r$). Decompose $z' = x' + iy'$ into real and imaginary part and define

$$R'\colon T' \to Q, \quad R(z) = e^{\pi i\varphi}\left(\frac{x'}{\|x'\|} + i\frac{y'}{\|y'\|}\right).$$

Here the points of Q (5.5.5) are denoted by $u + iv$. (Observe that R' does not depend on the choice of φ.) Then

$$R\colon T' \to Q \simeq F$$

is the composition of R' with the diffeomorphism (5.5.6).

§6. THE PICARD–LEFSCHETZ FORMULAS

6.1. Let $f\colon Y \to G$ be as in 5.1. When the singular values $t_1, \ldots, t_r$ of $f\colon Y \to G$ are removed from G,

$$G^* = G \setminus \{t_1, \ldots, t_r\}, \tag{6.1.1}$$

and the corresponding singular fibres are removed from Y,

$$Y^* = Y \setminus f^{-1}\{t_1, \ldots, t_r\}, \tag{6.1.2}$$

a locally trivial fibre bundle $f\colon Y^* \to G^*$ with typical fibre $Y_b \simeq X_b$ remains according to Ehresmann's fibration theorem. The fundamental group $\pi_1(G^*, b)$ acts on the homology of Y_b. This action is called the *monodromy* of $f\colon Y \to G$. It will be studied in §6 and §7. The main results are the Picard–Lefschetz formula (6.3.3) which holds in general and the semi-simplicity of the monodromy (7.3.3) which holds in the special situation described in §1.2 above.

Let t be a local coordinate of G in a neighbourhood of t_ν. Choose $\rho > 0$ so small that the disk D_ν with centre t_ν and radius ρ does not meet any t_μ, $\mu \neq \nu$. Let l_ν be any path in G^* from b to $t_\nu + \rho$ and let

$$\omega_\nu(s) = t_\nu + \rho e^{2\pi is}, \quad 0 \leq s \leq 1, \tag{6.1.3}$$

be the path which encircles t_ν once. Then

$$w_\nu = l_\nu^{-1} \cdot \omega_\nu \cdot l_\nu \tag{6.1.4}$$

is called an elementary path encircling t_ν, see Fig. 3 and also Fig. 1 in §5.3.

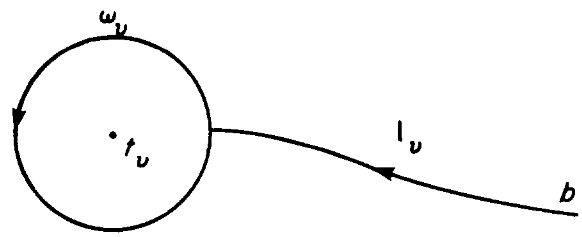

Fig. 3

The fundamental group $\pi_1(G^*, b)$ is a free group generated by the homotopy classes $[w_1], \ldots, [w_r]$ of the elementary paths. If the t_ν are suitably ordered and the l_ν are suitably chosen there is exactly one relation

$$[w_1] \cdot [w_2] \cdots [w_r] = 1.$$

The Picard–Lefschetz formula describes the action of the elementary paths w_i on $H_q(Y_b)$. It requires special elements of $H_{n-1}(Y_b)$ and $H_n(Y_+, Y_b)$ which will now be defined using the results of §5.

6.2. Consider the following sequence of homomorphisms induced by inclusions

$$(6.2.1)\quad H_n(T, F) \xrightarrow[(5.4.1)]{\cong} H_n(T_i, F_i) \xrightarrow[(5.3.2)]{\text{mono}} H_n(Y_+, L) \xrightarrow[(5.3.1)]{\cong} H_n(Y_+, Y_b).$$

According to (5.5.9) an orientation of the disk Δ determines a generator $[\Delta]$ of $H_n(T, F)$. The monomorphism (6.2.1) transforms $[\Delta]$ into an element $\Delta_i \in H_n(Y_+, Y_b)$. The elements $\Delta_1, \ldots, \Delta_r$ generate $H_n(Y_+, Y_b)$ freely. The connecting homomorphism $\partial_*: H_n(Y_+, Y_b) \to H_{n-1}(Y_b)$ transforms Δ_i into

$$(6.2.2)\qquad \delta_i = \partial_* \Delta_i \in H_{n-1}(Y_b), \qquad i = 1, \ldots, r.$$

Lefschetz [L] Chap. II, §13 and Chap. V, §6, calls δ_i a *vanishing cycle* and Δ_i the corresponding *thimble*: The geometric boundary $\partial\Delta = S^{n-1} \subset F \subset F_i$ is an embedded $(n-1)$-sphere in F_i (see, 5.5.7). Since the inverse image $f^{-1}(l_i)$ is trivially fibred there is an embedding

(6.2.3) j: $F_i \times l_i \to Y$, $j(F_i \times l_i) = f^{-1}(l_i)$, $j(y, t_i + \rho) = y$ *and* $f \circ j(y, \lambda) = \lambda$ *for* $y \in F_i$ *and* $\lambda \in l_i$

Then the thimble

$$(6.2.4)\qquad C_i = \Delta \cup j(S^{n-1} \times l_i)$$

represents Δ_i. Its boundary ∂C_i is an embedded $(n-1)$-sphere in Y_b, which represents δ_i: see Fig. 4.

When the sphere C_i is pushed along the thimble from Y_b following l_i into $F_i = Y_{t_i+\rho}$ and further into the singular fibre Y_{t_i} it vanishes at the critical point x_i, hence the name "vanishing cycle".

6.3. A tubular neighbourhood of S^{n-1} in F_i is F, and S^{n-1} lies in F as the zero section Q_0 lies in the tangent bundle Q of the $(n-1)$-sphere, see (5.5.5)–(5.5.7). The

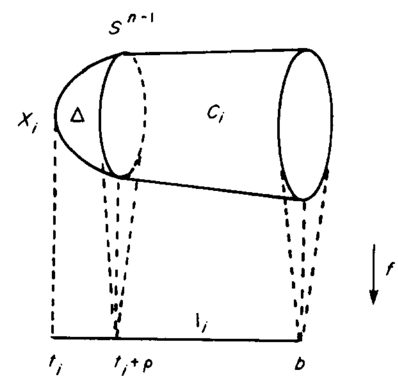

Fig. 4

self-intersection number of Q_0 in Q (i.e. the Euler number of S^{n-1}) is known to be 0 or 2 depending on whether n is even or odd. This number is calculated with respect to the usual orientation of Q (first an orientation of Q_0 and then the corresponding orientation of a fibre). The complex structure of F induces another orientation of Q. It differs from the usual one by the factor $(-1)^{(n-1)(n-2)/2}$ and hence the self-intersection number of S^{n-1} in F_i is $(-1)^{(n-1)(n-2)/2}(1-(-1)^n)$. The orientation preserving diffeomorphism

(6.3.1) $$h_i\colon F_i \simeq Y_b, \quad h_i(y) = j(y, b), \quad y \in F_i,$$

maps S^{n-1} onto C_i. Hence:

(6.3.2) *The normal bundle of the vanishing cycle C_i in Y_b is isomorphic to the tangent bundle of the $(n-1)$-sphere. The self-intersection number is*

$$\langle \delta_i, \delta_i \rangle = \begin{cases} 0, & n \text{ even} \\ (-1)^{(n-1)/2} \cdot 2, & n \text{ odd} \end{cases}$$

(6.3.3) THE PICARD–LEFSCHETZ FORMULA. *If $q \neq n-1$ the fundamental group $\pi_1(G^*, b)$ acts trivially on $H_q(Y_b)$. For $q = n-1$ the elementary path w_i, see* (6.1.3) *and* (6.1.4), *acts by*

$$(w_i)_*(x) = x + (-1)^{n(n+1)/2}\langle x, \delta_i \rangle \delta_i, \quad x \in H_{n-1}(Y_b).$$

For $n = 2$ the formula (6.3.3), up to the coefficient $\langle x, \delta_i \rangle$, is due to Picard [12, Tome I, p. 95]. The coefficient $\langle x, \delta_i \rangle$ was first obtained by Lefschetz. In his book [L] (6.3.3) is the "théorème fondamentale", Chap. II, §9, upon which he builds the investigation of algebraic surfaces. Later in [L], Chap. V, Nos. 6 and 7, he generalizes the result from surfaces to higher dimensional manifolds. The following sections contain the proof of (6.3.3).

6.4. This section contains topological preliminaries. Let $f\colon A \to B$ be a continuous mapping and $B^* \subset B$ a subspace such that f fibres $E = f^{-1}(B^*)$ locally trivially over B^*. The fibre over $y \in B$ is denoted by $F_y = f^{-1}(y)$. Let $w\colon I = [0, 1] \to B^*$ be a path from $a = w(0)$ to $b = w(1)$. The induced bundle w^*E over I is trivial, in other words: There is a continuous mapping

(6.4.1) $$W\colon F_a \times I \to E \hookrightarrow A$$

with the following properties:

$f \circ W(x, t) = w(t)$ and $W(x, 0) = x$ for $x \in F_a$, $t \in I$.
For any fixed $t \in I$ W_t: $F_a \simeq F_{w(t)}$, $x \mapsto W(x, t)$, is a homeomorphism; for any L with $F_a \cup F_b \subset L \subset A$ the lifting W is a mapping between pairs

$$W: F_a \times (I, \partial I) \to (A, L).$$

The homotopy class of the path w determines W up to homotopy relative to ∂I and L and determines W_1: $F_a \simeq F_b$ up to isotopy. Since the induced isomorphism in homology $(W_1)_*$ depends only on w, it will be denoted by

(6.4.2)
$$w_* = (W_1)_*: H_*(F_a) \simeq H_*(F_b).$$

If w is closed, W_1 is called a *geometric monodromy* and w_* the *algebraic monodromy* along w. Let

$$\iota \in H_1(I, \partial I)$$

be the canonical generator. Then

(6.4.3)
$$\begin{aligned} \tau_w: H_q(F_a) &\longrightarrow H_{q+1}(F_a \times (i, \partial I)) \xrightarrow{W_*} H_{q+1}(A, L) \\ x &\mapsto x \times \iota \end{aligned}$$

is called the *extension* along w. It depends only on the homotopy class of w. Further properties of the extension are:

(6.4.4) *If $L \supset f^{-1}$ (image of w), then $\tau_w = 0$.*

(6.4.5) Naturality. *A commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & A_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ B & \xrightarrow{\varphi} & B_1 \end{array}$$

with $\varphi(B^*) \subset B_1^*$ and $\varphi(L) \subset L_1$ induces the commutative diagram

$$\begin{array}{ccc} H_q(F_a) & \xrightarrow{(\Phi_a)_*} & H_q(F_{1,\varphi(a)}) \\ {\scriptstyle \tau_w}\downarrow & & \downarrow{\scriptstyle \tau_{\varphi \circ w}} \\ H_{q+1}(A, L) & \xrightarrow{\Phi_*} & H_{q+1}(A_1, L_1) \end{array}$$

(6.4.6) *If ∂_*: $H_{q+1}(A, L) \to H_q(L)$ denotes the connecting homomorphism, then*

$$(-1)^q \partial_* \tau_w(x) = w_*(x) - x, \qquad x \in H_q(F_a).$$

Here the image of x under $F_a \to L$ is also denoted by x, and similarly for $w_*(x)$.

(6.4.7) COMPOSITION. *If w is a path from a to b and v is a path from b to c and if $L \supset F_a \cup F_b \cup F_c$ then*

$$\tau_{v \circ w} = \tau_v \circ w_* + \tau_w \quad \textit{and} \quad (v \circ w)_* = v_* \circ w_*.$$

A relative version is also needed: Let $A' \subset A$ be a subspace, denote

$$E' = E \cap A' \quad \text{and} \quad F'_y = F_y \cap A'.$$

Assume: (1) f fibres the pair (E, E') locally trivially over B^* and (2) F'_a is a strong deformation retract of A'. Then

$$W\colon (F_a, F'_a) \times (I, \partial I) = (F_a \times I, F_a \times \partial I \cup F'_a \times I) \to (A, L \cup A')$$

and the *relative extension* is defined to be

$$\text{(6.4.8)} \qquad \begin{array}{ccc} \tau_w\colon H_q(F, F'_a) \to H_{q+1}((F_a, F'_a) \times (I, \partial I)) & \xrightarrow{W_*} & H_{q+1}(A, L \cup A') \\ x \mapsto x \times \iota & & \uparrow \text{inc}_* \\ & & H_{q+1}(A, L) \end{array}$$

Mutatis mutandis the results (6.4.4)–(6.4.7) remain true in the relative case.

The extension along the elementary paths (6.1.4) will now be calculated. The procedure is the same as in §4 but in reversed order.

6.5 First the situation of §5.5. is studied, $f\colon T \to D = \{t \in \mathbf{C} \,|\, |t| \le \rho\}$, $f(z) = z_1{}^2 + \cdots + z_n{}^2$, with $D^* = D\backslash 0$, typical fibre F. This is a relative situation due to T' (5.4.2). Both assumptions for (6.4.8) are fullfilled, (1) because of the relative version of Ehresmann's fibration theorem and (2) because of (5.4.2). Therefore the relative extension

$$\tau_\omega\colon H_{n-1}(F, F') \to H_n(T, F)$$

along the path $\omega\colon I \to D\backslash 0$, $\omega(t) = \rho e^{2\pi i t}$, is defined. The other dimensions $\neq n$ are uninteresting because then the homology of (T, F) vanishes (5.5.9). Let $s = \partial_*[\Delta] = [S^{n-1}] \in H_{n-1}(F)$. Choose $c \in H_{n-1}(F, F')$ so that $\langle c, s\rangle = 1$. Then

$$\text{(6.5.1)} \qquad \tau_\omega(c) = -(-1)^{n(n-1)/2}[\Delta].$$

This is the main result of this section. In its proof explicit geometric considerations will play an essential rôle. Since $H_n(T, F)$ is generated by $[\Delta]$, "$\tau_\omega(c) = \gamma \circ [\Delta]$ with $\gamma \in \mathbf{Z}$" is obvious. It remains to prove

$$\text{(6.5.2)} \qquad \gamma = -(-1)^{n(n-1)/2}.$$

For this purpose the following diagram is considered:

$$\begin{tikzcd}
H_n(F \times I, \partial(F \times I)) \arrow[r, "W_*"] \arrow[d, "\partial_*", "\simeq"'] & H_n(T, T' \cup F) \arrow[r, "\text{inc}_*", "\simeq"'] \arrow[d, "\partial_*", "\simeq"'] & H_n(T, F) \arrow[d, "\partial_*"] & \\
H_{n-1}(\partial(F \times I)) \arrow[r, "W_*"] \arrow[d, phantom, "\wr\wr", "(5.5.6)"'] & H_{n-1}(T' \cup F) \arrow[r, "R_*", "\simeq"'] \arrow[d, phantom, "(*)"] & H_{n-1}(F) \arrow[r, "(Re)_*", "\simeq"'] & H_{n-1}(S^{n-1}) \arrow[d, phantom, "\wr\wr", "(5.5.6)"'] \\
H_{N-1}(\partial(Q \times I)) \arrow[rrr, "g_*"] & & & H_{n-1}(Q_0) \arrow[d, equal] \\
H_{n-1}(\partial(C \times I)) \arrow[rrr, "g_*"] \arrow[u, "\text{inc}_*", "\simeq"'] & & & H_{n-1}(Q_0).
\end{tikzcd}$$

In the diagram the following spaces and mappings occur:

$W: F \times I \to T,\ (x, t) \mapsto e^{\pi i t} \cdot z$
$R: T' \cup F \to F,\ R|F = id_F$ and for $R|T'$ see the end of §5.5.
Re = real part
Q and Q_0 as in §5.5, $Q' = \{(u, v) \in Q \mid \|v\| = 1\}$
$g: \partial(Q \times I) = Q' \times I \cup Q \times \partial I \to Q_0,\ (u + iv, t) \to Re(e^{i\pi t}(u + iv))$
$C = \{e_1 + iv \mid v \in \mathbf{R}^n,\ v \perp e_1\}$, where $e_1 = (1, 0 \ldots 0) \in \mathbf{R}^n$

In the following other unit vectors e_ν will also occur.

All partial diagrams commute; this is mostly obvious, with (*) it must be checked by comparing the mappings $Re \circ R \circ W$ and g explicitly. Starting from $c \times \iota \in H_n(F, \partial(F \times I))$ the upper line of the diagram yields $\tau_\omega(c) = \gamma \cdot [\Delta]$. The isomorphisms of the right boundary transform this element into $\gamma \cdot [Q_0]$. Here $Q_0 = \{(u, 0) \in \mathbf{R}^n \times \mathbf{R}^n \mid \|u\| = 1\}$ is oriented as the unit sphere of the canonically oriented $\mathbf{R}^n$. The commutativity of the diagram implies that the isomorphisms of the left boundary applied to $c \times \iota$ followed by g_* yield $\gamma \cdot [Q_0]$, too. In order to determine γ two things must be checked: Which orientation of $\partial(C \times I)(\simeq S^{n-1})$ is determined by $c \times \iota$, and: What is the mapping degree of $g: \partial(C \times I) \to Q_0$?

(6.5.3) The orientation of the coordinate system $(v_2, \ldots, v_n)$ on C differs from the orientation which $c \in H_{n-1}(F, F')$ determines by the factor

$$(-1)^{n(n-1)/2}.$$

This is proved by considering a neighbourhood of e_1 in F. Here $(v_2, \ldots, v_n)$ followed by the positively oriented coordinate system $(u_2, \ldots, u_n)$ of Q_0 form the coordinate system $(v_2, \ldots, v_n, u_2, \ldots, u_n)$ of F. Since $\langle c, s \rangle = 1$ the orientation of $(v_2, \ldots, v_n)$ differs from the orientation of c by the same factor as the orientation of $(v_2, \ldots, v_n, u_2, \ldots, u_n)$ differs from the canonical orientation of F. The latter is determined by any complex coordinate system, e.g. by $(u_2 + iv_2, \ldots, u_n + iv_n)$ which yields the positively oriented real system $(u_2, v_2, \ldots, u_n, v_n)$. Its orientation differs from the one of $(v_2, \ldots, v_n, u_2, \ldots, u_n)$ by the sign of the corresponding coordinate permutation, i.e. by $(-1)^{n(n-1)/2}$.

The degree of $g: \partial(C \times I) \to Q_0$ is calculated in the following way: The point $\left(e_1 + ie_2, \frac{1}{2}\right) \in \partial C \times I \subset \partial(C \times I)$ is the only inverse image point of $-e_2 \in Q_0$. Therefore γ equals the local mapping degree of g at $\left(e_1 + ie_2, \frac{1}{2}\right)$. The orientation of C given by $(v_2, \ldots, v_n)$ followed by the canonical orientation of I determines an orientation of $C \times I$ and hence of $\partial(C \times I)$. With respect to this orientation $(v_3, \ldots, v_n, t)$ is a positively oriented coordinate system of $\partial(C \times I)$ in a neighbourhood of $\left(e_1 + ie_2, \frac{1}{2}\right)$. In a neighbourhood of $-e_2$ in Q_0 the positively oriented coordinate system $(u_1, u_3, \ldots, u_n)$ is chosen. With respect to these coordinates $g(v_3, \ldots, v_n, t) = (\cos \pi t, -\sin \pi t \cdot v_3, \ldots, -\sin \pi t \cdot v_n)$. The Jacobian of this system at $\left(e_1 + ie_2, \frac{1}{2}\right)$ is negative; hence with respect to these orientations the degree of g equals -1. This together with (6.5.3) yields γ as in (6.5.2).

6.6. Following the procedure of §5 in reversed order $f\colon T_\nu \to D_\nu$ as defined by (5.2.1) must be considered next. Here $D_\nu^* = D_\nu \backslash t_\nu$ and $t_\nu + \rho$ is the base point. The (absolute) extension along the path ω_ν (6.1.2) is

$$(6.6.1) \qquad \tau_{\omega_\nu}\colon H_{n-1}(F_\nu) \to H_n(T_\nu, F_\nu), \quad x \mapsto -(-1)^{n(n-1)/2}\langle x, s\rangle \cdot [\Delta].$$

Here $H_n(T_\nu, F_\nu)$ is freely generated by $[\Delta]$ according to (5.4.1) and (5.5.9) and $s = \partial_*[\Delta] \in H_{n-1}(F_\nu)$. As in 5.4 this formula is proved by reduction to the case (T, F). Let $r\colon (F_\nu, \emptyset) \hookrightarrow (F_\nu, F_\nu \backslash \mathring{B})$ be the inclusion. Because of (5.4.2) the relative extension $\tau_{\omega_\nu}\colon H_{n-1}(F_\nu, F_\nu \backslash \mathring{B}) \to H_n(T_\nu, F_\nu)$ is also defined. The naturality of the extension (6.4.5) makes the following diagram commutative:

$$\begin{tikzcd}
H_{n-1}(F_\nu) \arrow[r, "r_*"] \arrow[dr, "\tau_{\omega_\nu}"'] & H_{n-1}(F_\nu, F_\nu \backslash \mathring{B}) \arrow[d, "\tau_{\omega_\nu}"] & H_{n-1}(F, F') \arrow[l, "\mathrm{exc}_*"', "\approx"] \arrow[d, "\tau_{\omega_\nu}\ (6.5.1)"] \\
 & H_{n-1}(T_\nu, F_\nu) & H_n(T, F). \arrow[l, "(4.4.1)"', "\approx"]
\end{tikzcd}$$

The homomorphism of the top line transforms $x \in H_{n-1}(F_\nu)$ into $\langle x, s\rangle \cdot c \in H_{n-1}(F, F')$. The desired result (6.6.1) follows now from (6.5.1) applied to c.

6.7. Still following the procedure of §5 in reversed order the starting point $f\colon Y_+ \to D_+$ with $D_+^* = D_+ \backslash \{t_1, \ldots, t_r\}$ is now reached: The extension along the elementary path w_i (6.1.4) will be calculated. Using the notation of 6.2 the result is

$$(6.7.1) \qquad \tau_{w_i}\colon H_{n-1}(Y_b) \to H_n(Y_+, Y_b), \quad x \mapsto -(-1)^{n(n-1)/2}\langle x, \delta_i\rangle \cdot \Delta_i.$$

In order to prove (6.7.1) consider the following diagram

$$\begin{tikzcd}
H_{n-1}(Y_b) \arrow[rr, "(l_i)_*"] \arrow[d, "\tau_{w_i}"'] \arrow[dr, "\tau_{w_i}"] & & H_{n-1}(F_i) \arrow[d, "\tau_{\omega_i}"] \arrow[dl, "\tau_{\omega_i}"'] \\
H_n(Y_+, Y_b) \arrow[r, "\approx"'] & H_n(Y_+, L) & H_n(T_i, F_i). \arrow[l]
\end{tikzcd}$$

Both lower triangles are commutative because the extension is natural (6.4.5). Using (6.4.7) $\tau_{w_i}\colon H_{n-1}(Y_b) \to H_n(Y_+, L)$ is calculated as follows: $\tau_{w_i} = \tau_{l_i^{-1}\omega_i l_i} = \tau_{l_i^{-1}} \circ \omega_{i*} \circ l_{i*} + \tau_{\omega_i} \circ l_{i*} + \tau_{l_i}$. The first and third summands are zero because image $l_i \subset L$, see (6.4.4). Thus $\tau_{w_i} = \tau_{\omega_i} \circ l_{i*}$ remains, i.e. the upper triangle of the diagram commutes, too. The result (6.7.1) follows now from (6.6.1).

If (6.4.6) is applied to (6.7.1) the Picard–Lefschetz formula (6.3.3) follows immediately.

§7. THE MONODROMY

7.1. In the previous §6 the monodromy has been introduced and studied for an arbitrary meromorphic function $f\colon Y \to G$ with non-degenerate critical values as described in §5.1. This investigation will now be continued for the more special situation of §1.2: Here Y is the modification of a projective manifold X along the axis of a pencil of hyperplanes $\{H_t\}_{t \in G}$, and f assigns to every $y \in Y$ the hyperplane H_t

through y. For the regular value $b \in G$ the hyperplane section $X_b = X \cap H_b$ and the fibre $Y_b = f^{-1}(b)$ will be identified: see (1.2.3).

7.2. *The module $I \subset H_{n-1}(Y_b)$ of invariant cycles as defined in* §3.9 *is exactly the submodule of those elements of $H_{n-1}(Y_b)$ which are invariant under the action of $\pi_1(G^*, b)$.*

This justifies the name "invariant". The proof is a combination of known facts: The homotopy classes of the elementary paths $w_1, \ldots, w_r$ generate $\pi_1(G^*, b)$ according to §6.1. Therefore $y \in H_{n-1}(Y_b)$ is invariant under the action of π_1 if and only if

$$y = w_{i*}(y) = y \pm \langle y, \delta_i \rangle \delta_i, \quad \text{i.e. } \langle y, \delta_i \rangle = 0 \quad \text{for } i = 1, \ldots, r.$$

Here the Picard–Lefschetz-formula (6.3.3) has been used. On the other hand $I = \{y \mid \langle y, x \rangle = 0 \textit{ for every } x \in V\}$ according the (3.9.5). Since V is generated by $\delta_1, \ldots, \delta_r$ (see §3.8 and §6.2), $I = \{y \mid \langle y, \delta_i \rangle = 0 \textit{ for } i = 1, \ldots, r\}$ and the result follows.

7.3. The main result of this §7 is the

MONODROMY THEOREM. *For coefficients in a field the following results are equivalent*:

(7.3.1) *The Hard Lefschetz Theorem, i.e. the equivalent results* (4.1.2)–(4.1.7).
(7.3.2) *$V = 0$ or V is a non-trivial simple π-module.*
(7.3.3) *$H_{n-1}(Y_b)$ is a semi-simple π-module.*

Here $\pi = \pi_1(G^*, b)$.

Proof that (7.3.2) implies (7.3.3): Consider the π-invariant submodule $I \cap V$ of V. Since V is simple, $I \cap V = 0$ or $= V$. The latter is impossible because π acts non-trivially on V and trivially on $I \cap V$: hence $I \cap V = 0$. This together with the dimension formula (3.9.6) shows that $H_{n-1}(Y_b) = I \oplus V$ is the direct sum of a trivial (hence semi-simple) and a simple π-module. Therefore $H_{n-1}(Y_b)$ itself is a semi-simple π-module.

Proof that (7.3.3) implies (4.1.7) and hence (7.3.1): The restriction of $\langle -, - \rangle$ to I is non-degenerate. Let $\check{I}$ denote the dual module of I. It suffices to show that $I \to \check{I}$, $z \mapsto \langle z, - \rangle$, is epimorphic: Let $\varphi \in \check{I}$ be given. Since $H_{n-1}(Y_b)$ is semi-simple I has a complementary π-invariant submodule $M \subset H_{n-1}(Y_b)$ so that $I \oplus M = H_{n-1}(Y_b)$. This makes it possible to extend φ to a linear form ψ on $H_{n-1}(Y_b)$:

$$\psi(x + y) = \varphi(x), \quad x \in I, \quad y \in M.$$

Since $\langle -, - \rangle$ is non-degenerate on $H_{n-1}(Y_b)$ there is exactly one $z \in H_{n-1}(Y_b)$ with $\langle z, - \rangle = \psi(-)$, i.e. there is exactly one $z \in H_{n-1}(Y_b)$ with

(7.3.4) $$\langle z, x + y \rangle = \varphi(x) \quad \text{for every } x \in I \quad \text{and } y \in M.$$

When z is replaced by αz, $\alpha \in \pi$, (7.3.4) remains true because $\langle \alpha z, x + y \rangle = \langle z, \alpha^{-1}(x + y) \rangle = \langle z, x + \alpha^{-1} y \rangle = \varphi(x)$. Since z is uniquely determined by (7.3.4) $z = \alpha z$ for every $\alpha \in \pi$, i.e. $z \in I$ and $\langle z, x \rangle = \varphi(x)$ for every $x \in I$.

Proof that (4.1.6), and hence (7.3.1), implies (7.3.2): Let $F \neq 0$ be a π-invariant submodule of V and $0 \neq x \in F$. Since by (4.1.6) $\langle -, - \rangle$ is non-degenerate on V and V is generated by the vanishing cycles $\delta_1, \ldots, \delta_r$ there is a δ_μ with $\langle x, \delta_\mu \rangle \neq 0$. Let w_μ be a corresponding elementary path according to the Picard–Lefschetz-formula (6.3.3): $w_{\mu *}(x) = x \pm \langle x, \delta_\mu \rangle \delta_\mu$. Therefore π acts non-trivially on x and δ_μ belongs to F. But then all vanishing cycles $\delta_1, \ldots, \delta_r$ and hence all of V are contained in F because of the following result:

(7.3.5) *If the coefficients are a field then for any two vanishing cycles δ_μ, δ_ν there is an $\alpha \in \pi$ with $\alpha \cdot \delta_\mu = \pm \delta_\nu$.*

The Monodromy Theorem and its proof have been adapted from [22] Exposé XVIII. The following sections are devoted to the proof of (7.3.5).

7.4. Let $X \subset \mathbf{P}_N$ be a hypersurface (possibly with singularities) and $G \subset \mathbf{P}_N$ a projective line in general position with respect to X, i.e. G avoids the singularities of X and intersects X transversally. Then $G \cap X = \{t_1, \ldots, t_r\}$ is finite and $r =$ *degree of* X. Choose a base point $b \in G \backslash X$.

(7.4.1) *The embedding $G \backslash X \hookrightarrow \mathbf{P}_N \backslash X$ induces an epimorphism of the fundamental groups.*

Let E be a projective subspace with $G \subset E \subset \mathbf{P}_N$. Then (7.4.1) implies that the embedding $E \backslash X \hookrightarrow \mathbf{P}_N \backslash X$ induces an epimorphism of the fundamental groups. Zariski in [21] proved even more: When $\dim E \geq 2$ and the position of E with respect to X is suitably general, $\pi_1(E \backslash X) \to \pi_1(\mathbf{P}_N \backslash X)$ is an isomorphism. Since Zariski's proof is not quite satisfactory, Hamm and Lê [8*a*] present a modern but rather long proof of Zariski's result. In order to make our presentation selfcontained here is a much shorter proof of the weaker result (7.4.1) following the ideas of Zariski:

All lines through b form a subspace $\check{\mathbf{P}}_{N-1}$ of the dual projective space $\check{\mathbf{P}}_N$. A base point $a \in \check{\mathbf{P}}_{N-1}$ is chosen so that the corresponding line is $G_a = G$. (In general the line through b which corresponds to $z \in \check{\mathbf{P}}_{N-1}$ is denoted by G_z.) The point b in $\mathbf{P}_N$ is blown up:

$$Q = \{(x, z) \in \mathbf{P}_N \times \check{\mathbf{P}}_{N-1} \mid x \in G_z\}.$$

Then there are two projections

$$\mathbf{P}_N \xleftarrow{\ p\ } Q \xrightarrow{\ f\ } \check{\mathbf{P}}_{N-1}.$$

The inverse image of b is

$$p^{-1}(b) = \{b\} \times \check{\mathbf{P}}_{N-1}.$$

The complement

$$p \colon Q \backslash p^{-1}(b) \simeq \mathbf{P}_N \backslash \{b\}$$

is mapped isomorphically. Let

$$Y = p^{-1}(X).$$

Since $b \notin X$, $p^{-1}(b) \cap Y = \emptyset$. The second projection $f: Q \to \check{\mathbf{P}}_{N-1}$ fibres Q locally trivially with typical fibre G. Let $C \subset \check{\mathbf{P}}_{N-1}$ consist of all lines through b which are not in general position with respect to X. This C is a proper algebraic subset, see 1.5. When C and $f^{-1}(C)$ are removed, the pair

$$Q^* = Q\backslash f^{-1}(C), \quad Y^* = Y\backslash f^{-1}(C)$$

is locally trivially fibred by f over $\check{\mathbf{P}}_{N-1}\backslash C$. This follows from the relative version of Ehresmann's fibration theorem because Y^* is smooth and $f|Y^*$ has maximal rank everywhere. Hence the difference $Q^*\backslash Y^*$ is fibred locally trivially over $\check{\mathbf{P}}_{N-1}\backslash C$ by f with typical fibre $G\backslash X$. The upper line of the following commutative diagram is part of the exact homotopy sequence of this fibration:

$$\begin{tikzcd}
\pi_1(G\backslash X, b) \arrow[r] \arrow[ddr, "i_*"'] & \pi_1(Q^*\backslash Y^*, (b, a)) \arrow[r, "f_*"] \arrow[d, "j_*"] & \pi_1(\check{\mathbf{P}}_{N-1}\backslash C, a) \\
& \pi_1(Q\backslash Y, (b, a)) \arrow[d, "p_*"] & \\
& \pi_1(\mathbf{P}_N\backslash X, b). &
\end{tikzcd}$$

This diagram shows. In order to show that i_* is epimorphic it suffices to find a counterimage $\beta \in \pi_1(Q^*\backslash Y^*)$ with $f_*(\beta) = 1$ for every $\alpha \in \pi_1(\mathbf{P}_N\backslash X)$. Now p_* and j_* are both epimorphic. For p_* this is shown most conveniently using a base point $b' \neq b$. Then every element in $\pi_1(\mathbf{P}_N\backslash X, b')$ is represented by a path which avoids b and such a path can (uniquely) be lifted to $Q\backslash Y$ because $p: Q\backslash(Y \cup p^{-1}(b)) \simeq \mathbf{P}_N\backslash(X \cup \{b\})$ is an isomorphism. Similarly it is shown that j_* is epimorphic: Since $p^{-1}(C) \cap (Q\backslash Y)$ has real codimension 2 every path in $Q\backslash Y$ can homotopically be deformed so that it avoids $p^{-1}(C)$ and thus is contained in $Q^*\backslash Y^*$. Let $\beta' \in \pi_1(Q^*\backslash Y^*)$ be an arbitrary counterimage of α, but eventually $f_*(\beta') \neq 1$. There is a path u in $\{b\} \times (\check{\mathbf{P}}_{n-1}\backslash C) \subset Q^*\backslash Y^*$ with $[f \circ u] = f_*\beta'$. Then $\beta = \beta'[u]^{-1}$ is a counterimage of α with $f_*(\beta) = 1$ because $p \circ j \circ u$ is constant.

7.5. Let $X \subset \mathbf{P}_N$ be a hypersurface (possibly with singularities) and let G_0 and G_1 be two lines in general position with respect to X which have the point $b \notin X$ in common (possibly $G_1 = G_2$). Let v_0 and v_1 be elementary paths in $G_0\backslash X$ (respectively $G_1\backslash X$) from and to b.

(7.5.1) *When X is irreducible the homotopy classes $[v_0]$ and $[v_1]$ are conjugate elements in $\pi_1(\mathbf{P}_N\backslash X, b)$.*

Proof. Let v_0 encircle the point $c_0 \in G_0 \cap X$ and v_1 encircle $c_1 \in G_1 \cap X$. The subset $Z \subset X$ consisting of all points x such that the line through b and x is not in general position is proper and algebraic. Since furthermore X is irreducible there is a path w in $X\backslash Z$ from c_0 to c_1. Let G_t be the line through b and $w(t)$, $0 \leq t \leq 1$. Choose the isomorphisms $\Phi_t: \mathbf{C} \simeq G_t\backslash\{b\}$ so that $\mathbf{C} \times [0, 1] \to \mathbf{P}_N$, $(x, t) \mapsto \Phi_t(z)$, is continuous. Let $w^*(t) = \Phi_t^{-1}(w(t))$. If ρ is sufficiently small the disk in G_t with centre $w(t)$ and radius ρ intersects X only in $w(t)$. Then $(t, s) \mapsto \Phi_t(w^*(t) + \rho \cdot e^{2\pi i s})$, $0 \leq s, t \leq 1$, is a free homotopy in $\mathbf{P}_N\backslash X$ between the paths $\omega_0(s) = w^*(0) + \rho \cdot e^{2\pi i s}$ and $\omega_1(s) = w^*(1) + \rho \cdot e^{2\pi i s}$, which encircle c_0 (respectively c_1) once. This implies that $v_0 = l_0^{-1}\omega_0 l_0$ and $v_1 = l_1^{-1}\omega_1 l_1$ are conjugate in $\pi_1(\mathbf{P}_N\backslash X, b)$.

7.6. *Proof of* (7.3.5). Let w_μ, w_ν be the elementary paths which belong to δ_μ and δ_ν. Let $\check{X}$ be the dual variety (see (1.4.1)). The homotopy classes $[w_\mu]$ and $[w_\nu]$ are conjugate in $\pi_1(\check{\mathbf{P}}_N\backslash\check{X})$ by (7.5.1), and since $\pi_1(G^*)\to\pi_1(\check{\mathbf{P}}_N\backslash\check{X})$ is epimorphic (7.4.1) there is a path u in G^* such that

$$(7.6.1) \qquad [u]\cdot[w_\mu]=[w_\nu]\cdot[u] \quad \text{in } \pi_1(\check{\mathbf{P}}_N\backslash\check{X}).$$

Consider the locally trivial fibre bundle p_2: $W\backslash p_2^{-1}(\check{X})\to\check{\mathbf{P}}_N\backslash\check{X}$ as in §2.3. The fibre bundle f^*: $Y^*\to G^*$ is obtained from it by restriction to $G^*\subset\check{\mathbf{P}}_N\backslash\check{X}$. Therefore the action of $\pi_1(G^*)$ on $H_{n-1}(Y_b)$ factors through $\pi_1(\check{\mathbf{P}}_N\backslash\check{X})$, and thus (7.6.1) implies that $u_*\circ w_{\mu *}=w_{\nu *}\circ u_*$. When this is applied to an arbitrary element $x\in H_{n-1}(Y_b)$ the Picard–Lefschetz-formula (6.3.3) yields

$$(7.6.2) \qquad \langle x,\delta_\mu\rangle u_*(\delta_\mu)=\langle u_*(x),\delta_\nu\rangle\delta_\nu.$$

The intersection form $\langle -,-\rangle$ is non-degenerate by Poincaré duality. Therefore either $\delta_\mu=0$, and hence $\delta_\nu=0$, or there is an element x such that $\langle x,\delta_\mu\rangle\neq 0$, i.e. $u_*(\delta_\mu)=c\cdot\delta_\nu$ with $c\in$ coefficient field. Then (7.6.2) implies $\langle u_*(x),\delta_\nu\rangle\delta_\nu=\langle u_*(x),u_*(\delta_\mu)\rangle u_*(\delta_\mu)=c^2\langle u_*(x),\delta_\nu\rangle\delta_\nu$; hence $c=\pm 1$.

§8. HOMOTOPY RATHER THAN HOMOLOGY

8.1. In 1957 Thom suggested that Lefschetz's theorem on the homology of hyperplane sections (3.6.1) could be proved quite differently using real Morse theory. This idea was elaborated in two papers by Andreotti–Frankel[1] and Bott[2]. The latter observed that this method even yields a better result, namely (using the notation of §3.6):

(8.1.1) *The pair* (X, X_b) *is* $(n-1)$-*connected.*

In his book[11, §7] Milnor presents Andreotti–Frankel's proof adapted to this stronger result.

The stronger version (8.1.1) of (3.6.1) yields of course stronger versions of the results in §3.7: In (3.7.1) the conclusion "$H_q(X, X\cap F)=0$ for $q\le n-1$" can be improved to "The pair $(X, X\cap F)$ is $(n-1)$-connected" and in (3.7.2) "$H_q(\mathbf{P}_N, Y)=0$ for $q\le n$" can be imprived to "$(\mathbf{P}_N, Y)$ is n-connected."

In the following sections it will be shown how Lefschetz's original method also yields the stronger result (8.1.1).

8.2. Two facts in homotopy theory will be used which I have not been able to find explicitly in the literature:

(8.2.1) *Let* (X, A) *and* (Y, B) *be* r- *respectively* s-*connected relative* CW-*complexes with finitely many cells. Then* $(X,A)\times(Y,B)=(X\times Y, X\times B\cup A\times Y)$ *is* $(r+s+1)$-*connected.*

Proof. If $X\backslash A$ has no cells in dimensions less than $r+1$ and $Y\backslash B$ has no cells in dimensions less than $s+1$, $X\times Y\backslash(X\times B\cup A\times Y)$ has no cells in dimensions less than $r+s+2$. Hence $(X,A)\times(Y,B)$ is $(r+s+1)$-connected. The general case is reduced to this special case in the following way: By attaching finitely many cells to A and X a new relative CW-complex (X', A') is obtained such that $X'\backslash A'$ has no cells in dimensions less than $r+1$ and such that (X', A') collapses to (X, A), see e.g. Switzer[15, 6.13]. Similarly (Y, B) is replaced by (Y', B'). Then $(X', A')\times(Y', B')$ is $(r+s+1)$-connected and it collapses to $(X, A)\times(Y, B)$ so that (8.2.1) follows in general.

(8.2.2) *Let $f:(X, A)\to(Y, B)$ be a relative homeomorphism. If (X, A) is an n-connected relative CW-complex, then (Y, B) is also n-connected.*

Proof. The relative CW-decomposition of (X, A) is mapped isomorphically onto a relative CW-decomposition of (Y, B). "Isomorphically" means: The mapping $e\mapsto f(e)$ is a dimension preserving bijection between the cells e of $X\backslash A$ and the cells of $Y\backslash B$ and, if χ is the characteristic mapping of e then $f\circ\chi$ is the characteristic mapping of $f(e)$. Since (X, A) is n-connected cells can be attached to A and X in such a way that the new relative CW-complex (X', A') collapses to (X, A) and $X'\backslash A'$ has no cells in dimensions less than $n+1$. Since (Y, B) has an isomorphic CW-decomposition cells can be attached in the same way to Y and B as to X and A. Then the new relative CW-complex (Y', B') collapses to (Y, B) and $Y'\backslash B'$ has no cells in dimensions less than $n+1$. Therefore (Y', B') is n-connected and hence so is (Y, B).

8.3. *Proof of* (8.1.1). By induction from $n-1$ to n: As in homology the beginning $n=1$ is trivial. Consider now the following sequence of pairs of spaces and continuous mappings which has occurred already in the definition of L', (3.6.5):

$$(X_b, X')\times(D_-, S^1)\overset{\Phi}{\cong}(Y_-, Y_0\cup Y'_-)\xrightarrow{\text{exc}}(Y, Y_+\cup Y')\xleftarrow{j}(Y, Y_b\cup Y')\xrightarrow{p}(X, X_b).$$

By the induction hypothesis (X_b, X') is $(n-2)$-connected. Therefore $(X_b, X')\times(D_-, S^1)$ is n-connected by (8.2.1). Since Φ is a homeomorphism the same holds true for $(Y_-, Y_0\cup Y'_-)$. The homotopy excision theorem, see, e.g. Switzer[15, 6.21], implies that $(Y, Y_+\cup Y')$ is also n-connected. As in homology the next step is the exact homotopy sequence where j_* occurs,

$$\text{(8.3.1)}\quad \cdots\longrightarrow\pi_q(Y_+\cup Y', Y_b\cup Y')\longrightarrow\pi_q(Y, Y_b\cup Y')\xrightarrow{j_*}\pi_q(Y, Y_+\cup Y')\longrightarrow\cdots.$$

In §8.4 below the following result, which should be compared to (3.2.2), will be proved:

(8.3.2) *The pair (Y_+, Y_b) is $(n-1)$-connected.*

Consider now the inclusions $(Y_+, Y_b)\hookrightarrow(Y_+, Y_b\cup Y'_+)\hookrightarrow(Y_+\cup Y', Y_b\cup Y')$. Since Y_b is a deformation retract of $Y_b\cup Y'_+$ the first inclusion induces isomorphisms of all homotopy groups. In particular $(Y_+, Y_b\cup Y'_+)$ is $(n-1)$-connected because of (8.3.2). The second inclusion is an excision, hence by the homotopy excision theorem $(Y_+\cup Y', Y_b\cup Y')$ is $(n-1)$-connected. Since $(Y, Y_+\cup Y')$ is also $(n-1)$-connected (even n-connected, as has been shown above), the exact sequence (8.3.1) implies that $(Y, Y_b\cup Y')$ is $(n-1)$-connected. Then (X, X_b) is also $(n-1)$-connected because p is a relative homeomorphism, and thus (8.2.2) can be applied.

8.4. *Proof of* (8.3.2). The contractibility of T (5.5.1) and the fact that F (5.5.2) has a $(n-1)$-sphere as deformation retract imply that (T, F) is $(n-1)$-connected. Since F

is a deformation retract of $T' \cup F$ (5.4.2), the pair $(T, T' \cup F)$ is also $(n-1)$-connected. Then the homotopy excision theorem, see e.g. Switzer[15, 6.21], is applied to $(T, T' \cup F) \to (T_i, T_i \backslash \mathring{B} \cup F_i)$, so that $(T_i, T_i \backslash \mathring{B} \cup F_i)$ is $(n-1)$-connected. Then (T_i, F_i) is $(n-1)$-connected because F_i is a deformation retract of $T_i \backslash \mathring{B} \cup F_i$ (5.4.2).

Let $k_i = l_i \cup D_i$, $K_i = f^{-1}(k_i)$ and $L_i = f^{-1}(l_i)$ (see Fig. 1 in §5.3). The $(n-1)$-connected pair (T_i, F_i) is a deformation retract of (K_i, L_i), and so the latter is also $(n-1)$-connected. Since Y_b is a deformation retract of L_i the pair (K_i, Y_b) is $(n-1)$-connected. This implies inductively that

(8.4.1) $\quad (K_1 \cup \cdots \cup K_r, Y_b)$ *is* $(n-1)$-*connected.*

The induction step from s to $s+1$ $(s = 1, \ldots, r-1)$ is as follows: The assumption "$(K_1 \cup \cdots \cup K_s, Y_b)$ is $(n-1)$-connected" implies because of the homotopy excision theorem that $(K_1 \cup \cdots \cup K_{s+1}, K_{s+1})$ is $(n-1)$-connected. The exact homotopy sequence of the triple

$$\pi_q(K_{s+1}, Y_b) \to \pi_q(K_1 \cup \cdots \cup K_{s+1}, Y_b) \to \pi_q(K_1 \cup \cdots \cup K_{s+1}, K_{s+1})$$

yields the $(n-1)$-connectivity of $(K_1 \cup \cdots \cup K_{s+1}, Y_b)$ and thus completes the induction step.

In (8.4.1) $K = K_1 \cup \cdots \cup K_r$ can be replaced by Y_+ because it is a deformation retract of Y_+ as has been observed in (5.3.1). This yields (8.3.2).

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