Vanishing cycles and perverse sheaves : Exercises session III

EPFL

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All sheaves are sheaves of $\mathbb{Q}$-vector spaces, all spaces are "nice" e.g complex algebraic varieties.

Exercise 1

Describe the stalks of $j_*\underline{\mathbb{Q}}_Y$ where $j : Y \to X$ is the inclusion if :

- $X = \mathbb{R}, Y = \{0\}$
- $X = \mathbb{R}, Y = \mathbb{R}\backslash\{0\}$
- $X = \mathbb{R}, Y = \mathbb{R}_{>0}$
- $X = \mathbb{R}^2, Y = \{(x, \sin(1/x)) : x > 0\}$

Find two sheaves F, G on a simply connected space X such that F, G have isomorphic stalks for all x but the sheaves F, G are not isomorphic.

Exercise 2

Prove that a local system of rank r is trivial if and only if $\dim \Gamma(X, \mathcal{L}) = r$.

Exercise 3

If $f : Y \to X$ is a continuous map, prove that $f^*\mathcal{L}$ is a local system on Y if $\mathcal{L}$ is a local system on X. Let $p : Y \to X$ be a covering space. If $\mathcal{L}$ is a local system on Y prove that $p_*\mathcal{L}$ is a local system.

Give an algebraic description of the functors p_*, p^* (in terms of representation theory of the fundamental groups $\pi_1(Y, y)$ and $\pi_1(X, x)$).

Exercise 4

If $\mathcal{L}$ is a local system on X, discuss when there is a finite covering $p : Y \to X$ such that $p^*\mathcal{L}$ is trivial.

Exercise 5

Let $j : U = \mathbb{C}^* \to \mathbb{C}$ be the inclusion and $\mathcal{L}$ is a local system on $\mathbb{C}^*$. Describe the sheaves $j_!\mathcal{L}, j_*\mathcal{L}$, i.e compute their stalks, their restriction to U and $\{0\}$ and their cohomology groups. Write down the answer if $\mathcal{L}$ has rank 1, and if $\mathcal{L}$ has rank 2 with monodromy matrix $T = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ where $\alpha \in \mathbb{Q}$. While here, calculate $i^? j^{?'} \mathcal{L}$ for $?, ?' \in \{*, !\}$.

Exercise 6 (harder)

Let $U = \mathbb{P}^1 \backslash \{z_1, z_2, z_3, z_4\}$. Let $\mathcal{L}$ be the local system with monodromy -1 at each of the z_i. Calculate $H^*(U, \mathcal{L})$.