

Vanishing cycles and perverse sheaves : Exercises session IV

EPFL

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All sheaves are sheaves of $\mathbb{Q}$ -vector spaces, all spaces are "nice" e.g complex algebraic varieties. All functors (f_* , etc) are derived.

Exercise 1

If $U = \mathbb{C}^* \subset X = \mathbb{C}$, let $\mathcal{F} = j_*\underline{\mathbb{Q}}_U$. Calculate i^*F , $i^!F$, i^*G and $i^!G$ where $G = \underline{\mathbb{Q}}_X$.
If $\mathcal{L}$ is a local system, calculate the stalks of $j_*\mathcal{L}$ and $j_!\mathcal{L}$. Check when these sheaves are isomorphic.

Exercise 2

Write the composition series for the perverse sheaves $j_*\underline{\mathbb{Q}}_U[1]$ and $j_!\underline{\mathbb{Q}}_U[1]$ where $U = \mathbb{C}^* \subset X = \mathbb{C}$.

Exercise 3

Show that up to isomorphism on $U = \mathbb{C}^*$ there is a unique indecomposable local system J_n of rank n with a map $J_n \rightarrow \underline{\mathbb{Q}}_U$. Calculate its decomposition factors into simple local systems. (This local system is important in Beilinson's construction of nearby cycles). Calculate the factors for $IC(U, J_n)$.

Exercise 4

If $\mathcal{L}$ has rank 1, compute the stalks of $IC(U, \mathcal{L})$. Calculate the composition series in the category of perverse sheaves.

Exercise 5

Let $d \in \mathbb{Z}$, $Z = \{0\}$, $U = \mathbb{P}^1 \setminus \{0\}$, and D^d be the category obtained by gluing $(D_Z^{\leq 0}, D_Z^{\geq 0})$ with $(D_U^{\leq d}, D_U^{\geq d})$. Prove that if $d < 0$ or $d > 2$, the heart $\mathcal{M}_d$ is a semisimple category.

Exercise 6

Let $U = \{(x_1, \dots, x_n) : x_i \in \mathbb{C}, x_1 x_2 \dots x_n \neq 0\}$. Calculate the stalks of $j_* \underline{\mathbb{Q}}_U$.

Exercise 7

Let $p : \mathbb{C}^2 \rightarrow \mathcal{N}$ be the map given by

$$(x, y) \mapsto \begin{pmatrix} xy & x^2 \\ -y^2 & -xy \end{pmatrix}$$

where $\mathcal{N}$ is the nilpotent cone for $\mathfrak{sl}_2$. Show that p restricts to a degree 2 covering map $\mathbb{C}^2 \setminus \{(0, 0)\} \rightarrow \mathcal{N} \setminus 0$. Calculate the stalks $j_* \underline{\mathbb{Q}}_U$ where $j : U = \mathcal{N} \setminus 0 \rightarrow \mathcal{N}$. Compare with the sheaf $j_* \underline{\mathbb{Z}}_U$.

Exercise 8

Let $X = \{M \in M_2(\mathbb{C}) : \det(M) = 0\}$ and $\tilde{X} = \{(L, M) \in \mathbb{P}^1 \times X : L \subset \ker(M)\}$. The natural map $f : \tilde{X} \rightarrow X$ is a resolution of singularities. Calculate the stalks of $f_* \underline{\mathbb{Q}}_X$, $j_* \underline{\mathbb{Q}}_U$ and ω_X .

Exercise 9

Calculate the stalks of the complex $IC(X)$ if $X = \{(x, y) \in \mathbb{C}^2 : xy = 0\}$.