Vanishing cycles and perverse sheaves : Exercises session I

EPFL

February 28, 2024

Exercise 0

Finish the proof of the diagram chasing to prove that $H_q(Y) \cong H_q(X) \oplus H_{q-2}(X')$ for every q.
Check the details about the Veronese embedding, and deduce that complete intersection have Betti numbers $1, 0, 1, 0, \dots$ except in middle dimension.

Exercise 1

Let $X = \mathbb{C}^2/\Lambda$ be a torus for a generic lattice Λ. Show that X does not admit a Lefschetz pencil by showing there are non holomorphic curve $C \subset X$.

Exercise 2

Let $f : X \to G$ be a Lefschetz fibration. Show that X is connected if and only if X_t is connected, where $t \in G$ is a regular value.
Show that there is a surjection $\pi_1(X_t) \to \pi_1(X)$ and that the kernel of this surjection is generated by conjugacy classes of vanishing cycles. (Hint : use the long exact sequence for homotopy groups of fibrations $F \subset E \to B$).

Exercise 3

Let $f : X \to G$ a Lefschetz pencil. If $\dim X = 2$, use the previous exercise to calculate the Betti numbers of X. (Hint : first calculate the Euler characteristic of X).

Exercise 4

Let $X \subset \mathbb{P}^n$ be a curve of genus g. If Y is the cone over X, calculate the cohomology groups of Y.

Exercise 5

Consider the Lefschetz pencil $a(x^3 + y^3 + z^3) - 3bxyz = 0$ in $\mathbb{P}^2$, with associated map $f : X \to G$ is given by

$$f(x, y, z) = \frac{x^3 + y^3 + z^3}{3xyz}$$

Calculate the base points of the system (written X' in the class), and the singular fibers.

Exercise 6

With same notation as in the previous exercise, calculate the vanishing cycles in $f^{-1}(0)$ corresponding to $1, j, j^2 \in \mathbb{P}^1$.
Hint : first check that $\Re(f^{-1}(0))$ form a topological circle. Extend it to a "thumble" inside $f^{-1}([0,1])$ by checking that $\Re(f^{-1}(t))$ is still topologically a circle, and finally check that $\Re(f^{-1}(1))$ has trivial homology).