COHOMOLOGY & BASE CHANGE I :

Definition: Let $f: X \to Y$ be a morphism of schemes, $\mathcal{F} \in QCoh(X)$ we say $\mathcal{F}$ is flat or flat over Y if $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$ module for all $x \in X$.

Remark: If $Y = \operatorname{Spec} A$ the above is equivalent to, $\forall \operatorname{Spec} B \subseteq X$, $\mathcal{F}|_{\operatorname{Spec} B} = \tilde{M}$ with $M \in \operatorname{FlatMod}(A)$.

Setup: $f: X \to Y$ proper map of Noeth. schemes. $\mathcal{F} \in Coh(X)$ flat over Y, A a Noetherian ring, $X_y :=$ fiber, $\mathcal{F}_y := \mathcal{F} \otimes \mathcal{O}_y k(y)$. $\Leftarrow$ this is actually $\mathcal{F}|_{X_y}$.

Goal: Study the cohomology group $H^i(X_y, \mathcal{F}_y)$ as $y \in Y$ varies, in particular we prove:
- $y \mapsto h^i(X_y, \mathcal{F}_y)$ is upper semi-continuous.
- $y \mapsto \chi(\mathcal{F}_y)$ is locally constant.

Appendix:

Theorem 0.1: $f: X \to Y$ proper, $\mathcal{F} \in Coh(X)$ then $\forall i > 0$ $(R^i f_*)(\mathcal{F}) \in Coh(Y)$

Lemma 0.2: Suppose $0 \to L_0 \to L_1 \xrightarrow{\partial_1} \_\_ \to L_n \to 0$ is an exact sequence of A modules such that L_i is flat $\forall i \geq 1$. Then L_0 is flat.

Proof: (Sketch) $0 \to \operatorname{Im}\partial_1 \to L_2 \to \_\_ \to L_n \to 0$, $0 \to L_0 \to L_1 \to \operatorname{Im}\partial_1 \to 0$ then take long exact sequence in $\operatorname{Tor}_A^i(M, -)$.

Corollary 0.3: Suppose $0 \to L_0 \to L_1 \to \_\_ \to L_n \to 0 = L^\bullet$ is an exact sequence of flat modules and $M \in \operatorname{Mod}_A$ then $L^\bullet \otimes_A M$ is exact.

Definition 0.4: Let $\phi: K^\bullet \to C^\bullet \in \operatorname{CoChain}(\operatorname{Mod}_A)$ then the mapping cone of ϕ is $L^\bullet$ with $L^i := K^i \oplus C^{i-1}$, the boundary map, $\partial^i = \begin{bmatrix} \partial^i_K & 0 \\ \phi & -\partial^{i-1}_C \end{bmatrix}$.

Theorem 0.5: ϕ is a quasi-isomorphism (i.e isomorphism of cohomology groups) $\Leftrightarrow L^\bullet$ acyclic.

Proof: $0 \to C' \to L^\bullet \to K^\bullet \to 0$, $C' = C^\bullet[-1]$ i.e $C'^p = C^{p-1}$, $\partial'^p = -\partial^{p-1}$.

Taking LES in cohomology: $\dots \to H^i(K^\bullet) \xrightarrow{\Phi^i} \underset{H^i(C^\bullet)}{H^{i+1}(C')} \to H^{i+1}(L^\bullet) \to H^{i+1}(K^\bullet) \to$

Theorem: Let $f, X, Y, \mathcal{F}$ be as in the setup, $Y = \operatorname{Spec} A$ affine. Then there exists a complex $K^\bullet$ finite complex of f.g. projective A-modules st $\forall A$ algebras B

$$H^i(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B) = H^i(K \otimes_A B).$$

Remark: Suppose $\mathcal{U} = \{U_i\}$ is a finite affine open cover for X. $\mathcal{U} \times_Y \operatorname{Spec} B = \{U_i \times_Y \operatorname{Spec} B\}$ is an open affine cover for $X \times_Y \operatorname{Spec} B$: $H^i(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B)$ can be computed as the Čech cohomology,

$$C^p(\mathcal{U} \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B) = \prod_{i_0 < \dots < i_p} \Gamma(U_{i_0, \dots, i_p} \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B)$$

$= \prod_{i_0 < \dots < i_p} \Gamma(U_{i_0, \dots, i_p}, \mathcal{F}) \otimes_A B = C^p(\mathcal{U}, \mathcal{F}) \otimes_A B$ so $H^i(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B) \cong H^i(\check{C} \otimes_A B)$.

Lemma I: Suppose $C^\bullet = 0 \to C_0 \to C_1 \to \_\_ \to C_n \to 0$ is a bounded complex of A-modules st $H^i(C^\bullet)$ are finitely generated. Then $\exists K^\bullet = 0 \to K_0 \to K_1 \to \_\_ \to K_n \to 0$ and a map $\phi: K^\bullet \to C^\bullet$

(i) $K^\bullet$ is a complex of f.g A-modules, for $i > 0$ K_i are free.

(ii) $H^i(\phi^i): H^i(K^\cdot) \xrightarrow{\sim} H^i(C^\cdot)$
(iii) If each C_i is flat so is K_0.

Lemma II: Let $\phi: K^\bullet \to C^\bullet$ be a quasi-iso of complexes of flat A-modules, then for every A algebra B we have $\phi \otimes B : K \otimes B \to C \otimes B$ is a quasi-isomorphism.
Proof: If $L^\bullet$ is the mapping cone for ϕ then $L^\bullet \otimes B$ is the mapping cone for $\phi \otimes B$. L^i is flat for all i and $L^\bullet$ is acyclic $\Rightarrow$ $L^\bullet \otimes B$ is exact $\Rightarrow$ $\phi \otimes B$ is a quasi-iso.

$$H^i(K^\bullet \otimes B) \cong H^i(C^\bullet \otimes B) \cong H^i(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B)$$

Definition: $\varphi: Y \to \mathbb{Z}_{\geq 0}$ is upper-semi-continuous if $\varphi^{-1}([r, +\infty))$ is closed.

Corollary: With $f, X, Y, \mathcal{F}$ as above,
(i) $\varphi: Y \to \mathbb{Z}_{\geq 0}$ $y \mapsto h^i(X_y, \mathcal{F}_y) = \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$ is upper semi-continuous.
(ii) $y \mapsto \chi(\mathcal{F}_y)$ is locally constant.
Proof: We can assume $Y = \operatorname{Spec} A$ is affine, $K^\cdot = 0 \to K_0 \xrightarrow{d} \_\_ \xrightarrow{d} K_n \to 0$, further assume that K_i are free. We have, $H^i(X_y, \mathcal{F}_y) \cong H^i(K^\bullet \otimes_A k(y)) \Rightarrow$

$$h^i(K \otimes_A k(y)) = \dim_{k(y)}(\ker d^i \otimes k(y)) - \dim_{k(y)}(\operatorname{Im} d^{i-1} \otimes k(y))$$
$$\parallel$$
constant $[\dim_{k(y)}(K_i \otimes k(y)) - \dim_{k(y)}(\operatorname{Im}(d^i \otimes k(y))) \Rightarrow \chi(\mathcal{F}_y) = \sum (-1)^i \dim_{k(y)}(K_i \otimes k(y))$

is constant. $\dim_{k(y)} \operatorname{Im}(d^i \otimes k(y))$ is upper semi-cont:

Let A be the matrix corresponding to the morphism of free modules $K^i \xrightarrow[A]{d^i} K^{i+1}$ taking Λ^n: $\Lambda^n K^i \xrightarrow[\Lambda^n A]{\Lambda^n d^i} \Lambda^n K^{i+1}$

$$\begin{array}{ccc} \Lambda^n K^i \otimes k(y) & \xrightarrow{\Lambda^n A \otimes k(y)} & \Lambda^n K^{i+1} \otimes k(y) \\ \parallel & & \parallel \\ \Lambda^n(K^i \otimes k(y)) & \xrightarrow{\Lambda^n(A \otimes k(y))} & \Lambda^n(K^{i+1} \otimes k(y)). \end{array}$$

if $\dim_{k(y)}(\operatorname{Im}(d^i \otimes k(y))) < n$ then
$\Lambda^n(A \otimes k(y)) = 0 \Leftrightarrow \operatorname{Im} k(y)\ A_{ij} = 0 \Rightarrow \{y \in Y \mid \dim_{k(y)} \operatorname{Im}(d^i \otimes k(y)) < r\}$ is closed.

Proof of Lemma I: We use reverse induction to construct $K^\bullet, \partial_K, \phi^i$ as follows:
Suppose for $i \geq m+1$ we constructed $K^i, \partial_K^i, \phi^i$

$$\begin{tikzcd} K^{m+1} \arrow[r, "\partial_K"] \arrow[d, "\phi^{m+1}"] & K^{m+2} \arrow[r, "\partial_K"] \arrow[d, "\phi^{m+2}"] & K^{m+3} \arrow[r] \arrow[d, "\phi^{m+3}"] & {} \\ C^{m+1} \arrow[r] & C^{m+2} \arrow[r] & C^{m+3} \arrow[r] & {} \end{tikzcd}$$

(i) $\partial^{i+1} \circ \partial^i = 0 \ \forall i \geq m+1$
(ii) $\partial^i_C \circ \phi^i = \phi^{i+1} \circ \partial^i_K$
(iii) $H^i(\phi): H^i(K^\cdot) \to H^i(C^\cdot)$ is an iso for $i \geq m+2$.
(iv) $\phi^{m+1}|_{\ker \partial^{m+1}} : \ker \partial^{m+1} \twoheadrightarrow H^{m+1}(C)$ is surjective.
Denote $\ker \phi^{m+1}|_{\ker \partial^{m+1}} = B^{m+1}$
As B^{m+1} is finitely generated, there exists a free module $K'^m \xrightarrow{\partial'} B^{m+1}$. There exists f.g free A-module $K''^m \xrightarrow{\lambda} H^m(C^\cdot)$ define $K^m = K'^m \oplus K''^m$. Since $\phi^{m+1} \circ \partial^m \subseteq \operatorname{Im} \partial^m_C$ and K' is free, $\phi^{m+1} \circ \partial^m$ lifts to $\phi': K'^m \to C^m$
Define $\partial^m = \partial' \oplus 0$ and $\phi = \phi' \oplus \phi''$
We constructed, $K^0 \to K^1 \to \_\_ \to K^n$ and define $K^{-n} = 0\ \forall n < 0$ and $K^0 = K^0 / \ker \partial^0 \cap \ker \phi^0$

$$\begin{tikzcd} K''^m \arrow[r, two heads, "\lambda"] \arrow[dr, bend left] \arrow[ddr, bend right, "\phi''"'] & H^m(C^\cdot) \\ & Z^m(C^\cdot) \arrow[u] \arrow[d, hook] \\ & C^m \end{tikzcd}$$

It remains to show K^0 is projective. Suppose $L^\cdot$ is the mapping cone of $\phi: K^\bullet \to C^\bullet$ then L^i is flat for $i > 0$ since ϕ is a quasi-iso $L^\bullet$ is acyclic $\Rightarrow L^0$ is flat and $L^0 = K^0 \oplus C^{-1} = K^0 \Rightarrow K^0$ is flat. $\square$

Remark: $\phi: A^\bullet \to B^\bullet$, $A \to B \to C(\phi) \to A[1]$ correspond to exact sequence in the derived category.