Setting: $/k=\bar{k}$, char k arbitrary, some useful notes: Edixhoven, van der Geer, Moonen

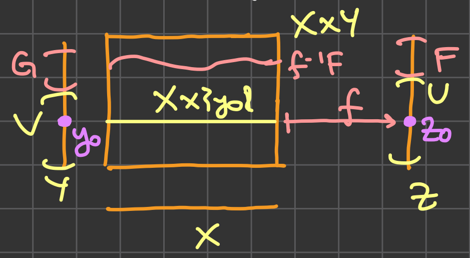
Definition (Recall): X variety$/k$ complete $m: X\times X\to X$, $i: X\to X$, $e: \operatorname{Spec} k\to X$

Today:
- $X(k)$ as a group: commutative, $n_X: X\to X$ $x\mapsto x^n$ is surj. for $p\nmid n$.
- Basic properties of Ω_X, X trivial, Ω_X trivial.

1. Commutativity:

Lemma (Rigidity Lemma): X, Y, Z varieties$/k$, X complete, $f: X\times Y\to Z$ st $\exists y_0\in Y$ and $z_0\in Z$ such that $f(X\times\{y_0\})=\{z_0\}$ $\Rightarrow$ $\exists g: Y\to Z$ $f=g\circ pr_Y$.

Picture:



Remark: Y, Z varieties $/k$ $a, b: Y\to Z$
$$\begin{tikzcd} Y \arrow[rr, "a,b"] \arrow[dr] & & Z \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$
if a and b agree on the closed points of some dense open then $a=b$.

Proof: $x_0\in X$, $(x_0, id): Y\to X\times Y$, $y\mapsto (x_0,y)$, if we want $f=g\circ pr_Y \Rightarrow f\circ(x_0,id)=g$. So define $g:=f\circ(x_0,id)$.
Let $U\subseteq Z$ be an open affine neigh. of z_0 and define $F:=Z\setminus U$, let $G=pr_Y(f^{-1}F)\subseteq Y$
$V=Y\setminus G$ notice that $y_0\in V$.
$\forall y\in V$ $f\circ(id,y)$ has image in U, indeed take by contradiction $x\in X$ st $f(x,y)\notin U$
$\Rightarrow (x,y)\notin f^{-1}(U)\Rightarrow y\in G$.
$f\circ id_y: X\longrightarrow U$ and as X proper $+$ U affine $\Rightarrow f\circ(id,y)$ constant equal to $g(y)$
$f(x,y)=g\circ pr_Y(x,y)$ for all closed $(x,y)\in X\times V$ $\Rightarrow f=g\circ pr_Y$. $\square$

Corollary 1: X, Y group var.$/k$ (complete), $f: X\to Y$ scheme morphism $\Rightarrow \exists h: X\to Y$ group var. morphism and $a\in Y$ st $f=L_a\circ h$ (here L_a is translation by a).

Proof: Up to replacing f by $L_{i(f(e_X))}\circ f$ we assume $f(e_X)=e_Y$. Consider the map
$\phi: X\times X\to Y$ $\quad\phi(x,x')=f(x.x').f(x')^{-1}.f(x)^{-1}$. But $\phi(X\times\{e_X\})=\{e_Y\}$ and
$\phi(\{e_X\}\times X)=\{e_X\}\Rightarrow$ by previous Lemma $\phi\equiv e_Y$ and f is a group morphism. $\square$

Corollary 2: X commutative.
Proof: $i: X\to X$ and $i(e_X)=e_X\Rightarrow i$ group morphism $\Rightarrow X$ commutative. $\square$

2. Ω_X Trivial (i.e $\cong\mathcal{O}_X^{\oplus \dim X}$):

Observation: X smooth. Indeed smooth locus is dense open, take $x_0\in X$ in the smooth locus, L_{x-x_0} gives $\mathcal{O}_{X,x}\cong\mathcal{O}_{X,x_0}\Rightarrow x\in X$ smooth for all x closed $\Rightarrow X$ smooth.

Setting: $k[\varepsilon]=k[x]/(x^2)$, $S=\operatorname{Spec} k[\varepsilon]$, X var$/k$, $X_\varepsilon=X\times_k S$, $X_\varepsilon=(|X|,\mathcal{O}_X[\varepsilon])$
Recall, $x\in X\Rightarrow T_{X,x}=\operatorname{Der}_k(\mathcal{O}_{X,x},k)$ and we can identify
$$T_{X,x}=\left\{\begin{tikzcd} & \operatorname{Spec} k \arrow[dl, "pt"'] \arrow[dr, "x"] & \\ S \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}\right\}$$
For $\tau: S\to X$ we get a derivation $\tau^{\#}: \mathcal{O}_{X,x}\to k[\varepsilon]$
$t\longmapsto t(x)+\tau(t)\varepsilon$.

Define $T_X:=\mathcal{H}om_{\mathcal{O}_X}(\Omega_X,\mathcal{O}_X)$. $U\subseteq X$ affine open $\Rightarrow T_X(U)=\operatorname{Der}_k(\mathcal{O}_X(U),\mathcal{O}_X(U))$
$$\cong\left\{\begin{tikzcd} & \mathcal{O}_X(U) & \\ \mathcal{O}_X(U) \arrow[ur, "id"] \arrow[rr] & & \mathcal{O}_X(U)[\varepsilon] \arrow[ul, "red"'] \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}\right\} \quad \text{taking Spec:} \quad \left\{\begin{tikzcd} & U \arrow[dl, "red"'] & \\ U_\varepsilon \arrow[rr] \arrow[dr] & & U \arrow[ul, "id"'] \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}\right\}$$
$t\longmapsto t+D(t).\varepsilon$

$\Rightarrow T_{X,x} = \mathcal{T}_{X,x} \otimes k(x)$, $\xi \in \mathcal{T}_X(U) \Rightarrow \xi : U_\varepsilon \to U \Rightarrow \xi \circ (x, id) : S \to U_\varepsilon \xrightarrow{\xi} U$

Proposition: X abelian variety, $\Rightarrow \exists$ natural iso $T_{X,0} \otimes \mathcal{O}_X \xrightarrow{\sim} \mathcal{T}_X$.

Proof: We want $\xi : T_{X,0} \to \mathcal{T}_X(X)$, take $\tau \in T_{X,0}$, $\tau : S \to X$

define, $\xi(\tau) = X_\varepsilon \xrightarrow{(id,\tau)} X \times X \xrightarrow{m} X$

Claim: ξ is k-linear (Exercise)

We obtain $\xi : T_{X,0} \otimes \mathcal{O}_X \to \mathcal{T}_X$ such that $\xi(U)(\tau \otimes s) = s . \xi(\tau)|_U$. To show that ξ is an iso it suffices that ξ is surjective.

Claim: $\xi(\tau)(x) = dL_{x,e}(\tau)$

Proof:

$$\begin{tikzcd}
S \arrow[r, "\tau"] \arrow[d, "{(x,id)}"'] \arrow[rrd, "\xi(\tau)(x)"] & X \arrow[d, "{(x,id)}"] \arrow[rd, bend left, "L_x"] & \\
X_\varepsilon \arrow[r, "{(id,\tau)}"] \arrow[rr, bend right, "\xi(\tau)"'] & X \times X \arrow[r, "m"] & X
\end{tikzcd}$$

Recall that for $f : Y \to Z$, $\tau : S \to Y \in T_{Y,y}$
$\Rightarrow df_y(\tau) = f \circ \tau$.
So $dL_{x,e}(\tau) = \xi(\tau)(x) \Rightarrow \xi(-)(x)$ is an iso
$\xrightarrow{\text{Nakayama}}$ ξ surj $\Rightarrow \xi$ iso. $\square$

Corollary: $\Omega \cong \mathcal{O}_X^{\oplus \dim X}$ and $\omega_X \cong \mathcal{O}_X$. Also every global vector field is translation invariant

3. n_X Surjective for $p \nmid n$:

- $$\begin{tikzcd}
X \arrow[r, "\Delta_n"] \arrow[rr, bend right, "n_X"'] & X^n \arrow[r, "m_n"] & X
\end{tikzcd}$$

Proposition: If $p \nmid n$ then n_X is surjective.

Proof: Let's determine $dm_{(0,0)}$

$$\begin{tikzcd}
0 \arrow[r, bend left] & T_{X,0} \arrow[l, bend left] \arrow[r, bend left, "d(id,0)_0"] & T_{X \times X,(0,0)} \arrow[l, bend left, "dpr_1(0,0)"] \arrow[r, bend left, "dpr_2(0,0)"] & T_{X,0} \arrow[l, bend left, "d(0,id)_0"] \arrow[r, bend left] & 0 \arrow[l, bend left]
\end{tikzcd}$$

$\Rightarrow T_{X \times X,(0,0)} \cong T_{X,0} \oplus T_{X,0}$
Take $d(id,0)_0(\tau_1) + d(0,id)_0(\tau_2)$

applying $dm_{(0,0)}$ to this element yields, $\tau_1 + \tau_2 \Rightarrow dm_{(0,0)} : T_{X,0}^{\oplus 2} \to T_{X,0}$
$(\tau_1, \tau_2) \mapsto \tau_1 + \tau_2$
iterating this shows, $dn_{X,0} =$ multiplication by n.
In particular $dn_{X,0}$ is surjective.
Suppose by contradiction n_X not surjective. $\Rightarrow n_X(X) \subsetneq X$ proper closed.
$X \to n_X(X) \to X$ Every irreducible component of $n_X^{-1}(0)$ has dimension
(dimension thm) $[\geq \dim X - \dim n_X(X) > 0$
$\dim n_X^{-1}(0) \geq 1$, $\iota : n_X^{-1}(0) \hookrightarrow X$ and $n_X \circ \iota = 0 \Rightarrow dn_X \circ d\iota_0 = 0 \Rightarrow T_{n_X^{-1}(0),0} = 0$
which is a contradiction $\Rightarrow n_X$ surj.