Abelian varieties

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$\boxed{\text{Recall}}$:

$\boxed{\text{Thm}}$ (Grothendieck): $f: X \to Y$ a proper morphism of Noeth. schemes, $Y = \operatorname{Spec} A$, $\mathcal{F}$ coherent on X, flat over Y. There exists a finite complex $K^\bullet$ of fin. gen. proj. A-modules and an iso ~~of~~ functors

$$H^p(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B) \simeq H^p(K^\bullet \otimes_A B) \quad (p \geq 0)$$

on the category of A-algebras.

Base change ~~of~~ theorems:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \arrow[dl, dotted] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

$g^*(R^p f_* \mathcal{F}) \to R^p f'_*(g'^*(\mathcal{F}))$ corresponds $g'^* R^p f_* \to R^p f'_* \circ g'^*$

$g'^* \dashv g'_* \rightsquigarrow id \to g'_* \circ g'^*$ unity

$R^p f_* \to R^p f_* \underbrace{g'_*}_{R^0 g'_*} g'_* \to g_* \circ R^p f'_* \circ g'^*$

(Grothendieck sp. seq. $(R^p f_*) \circ (R^q g'_*)(A) \Rightarrow R^{p+q}(f_* \circ g_*)(A)$

$\hookrightarrow R^p f_* R^0 g'_* g'^* \to R^p(f_* \circ g'_*) \circ g'^* = R^p(g_* \circ f'_*) \circ g'^*$

$\boxed{\text{Corollary}}\boxed{2}$: $f: X \to Y$ as in thm above (do not assume that Y affine) Y reduced & connected. $\forall p$, TFAE:

[(1) $y \in Y \longmapsto \dim_{k(y)} H^p(X_y, \mathcal{F}_y)$ constant]

[(2) $\overset{\mathcal{E}=}{R^p f_* \mathcal{F}}$ locally free, and moreover $R^p f_* \mathcal{F} \otimes_{\mathcal{O}_Y} k(y) \xrightarrow{\sim} H^p(X_y, \mathcal{F}_y)$]

If these conditions are satisfied $R^{p-1} f_* \mathcal{F} \otimes_{\mathcal{O}_Y} k(y) \xrightarrow{\sim} H^{p-1}(X_y, \mathcal{F}_y)$.

$\boxed{\text{Proof}}$:

2 ⇒ 1: $R^p f_* \mathcal{F} = \mathcal{E}$ locally free $\overset{Y \text{ conn.}}{\Longrightarrow}$ constant rank r (also finite)

take $y \in Y$. $\mathcal{E} \otimes k(y) \simeq H^p(X_y, \mathcal{F}_y)$, and around y: $\mathcal{E}|_V \cong \mathcal{O}_Y|_V^{\oplus r}$

$\dim_{k(y)} H^p(X_y, \mathcal{F}_y) = \dim_{k(y)} \mathcal{E}|_V \otimes k(y) = \dim_{k(y)} \mathcal{O}_Y|_V^{\oplus r} \otimes k(y) = \dim_{k(y)} k(y)^{\oplus r} = r.$

For $1 \Rightarrow 2$ we start with the following Lemma:

$\boxed{\text{Lemma 1}}$: Y-reduced, $\mathcal{F}$ coherent on Y, $\dim_{k(y)} \mathcal{F} \otimes_{\mathcal{O}_Y} k(y) = r$ $\forall y \in Y$

$\Rightarrow$ $\mathcal{F}$ locally free of rank r.

$\boxed{\text{Pf}}$: $Y = \operatorname{Spec} A$; let us take $y \in Y$, $\mathcal{F}_y$ finitely generated module, and $\dim_{k(y)} \mathcal{F}_y \otimes k(y) = r$; we take generators $\sigma_1, -, \sigma_r \in \mathcal{F}_y$ lifts of generators of $\mathcal{F}_y \otimes k(y)$. $\sigma_1, -, \sigma_r$ can be extended to small nbhd of y.

$\hookrightarrow$ $\sigma: \mathcal{O}_Y|_V^{\oplus r} \longrightarrow \mathcal{F}|_V$, σ_y surjective.

Now $\mathcal{O}_Y|_V^{\oplus r} \xrightarrow{\sigma} \mathcal{F}|_V \longrightarrow \operatorname{coker}(\sigma) \longrightarrow 0$. As $\operatorname{coker}(\sigma)_y = 0$, $\operatorname{coker}(\sigma)$ is 0 ~~in~~ a nbhd around y $\Rightarrow$ by shrinking V, we obtain $\operatorname{coker}(\sigma) = 0$. So σ surjective.

We want to show σ injective. Denote $\mathcal{I} = \ker(\sigma)$.

Consider $\sigma_{y'} \otimes k(y'): \mathcal{O}_{Y,y'}^{\oplus r} \otimes k(y') \longrightarrow \mathcal{F}_y \otimes k(y')$ for $y' \in V$.

This is an iso (surj. between vect. sp. of same finite dimension).

$\hookrightarrow$ we can conclude that $\mathcal{I}_{y'} \subset m_{y'} \mathcal{O}_{y'}^{\oplus r}$ $\forall y' \in V$.

Recall that $Y = \operatorname{Spec} B$ reduced $\Rightarrow$ B reduced.

If y' corresponds to p prime ideal of B. We have the following situation: $M \subseteq B^{\oplus r}$ s.t. $M_p \subseteq (pB_p) B_p^{\oplus r}$ $\forall p \in \operatorname{Spec} B$

$\hookrightarrow$ $M = 0$ $\Rightarrow$ $\mathcal{I} = 0$. $\square$(Lemma 1)

$\boxed{\text{Lemma 2}}$: Y reduced, Noeth., affine, $\mathcal{F} \xrightarrow{\phi} \mathcal{L}$ morphism of $\underbrace{\text{locally}}_{\text{coherent}}$ free sheaves on Y. If we have that $\dim(\operatorname{im}(\phi \otimes k(y)))$ locally constant $\Rightarrow$ $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$, $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$, $\phi = \begin{bmatrix} 0 & \text{isom} \\ 0 & 0 \end{bmatrix}$.

$\boxed{\text{Pf}}$: $\mathcal{F} \xrightarrow{\phi} \mathcal{L} \longrightarrow \mathcal{L}/_{\operatorname{im}\phi} \to 0$

$\hookrightarrow$ we want $\mathcal{L}/_{\operatorname{im}\phi}$ loc. free. We know $\dim(\operatorname{im}(\phi \otimes k(y)))$ loc. const.

$\mathcal{F} \otimes k(y) \longrightarrow \mathcal{L} \otimes k(y) \longrightarrow \underbrace{\mathcal{L}/_{\operatorname{im}\phi} \otimes k(y)}_{\cong \mathcal{L} \otimes k(y) / \operatorname{Im}(\phi \otimes k(y))} \longrightarrow 0$. $\Rightarrow$ $\dim(\mathcal{L}/_{\operatorname{im}\phi} \otimes k(y))$ loc. const.

$\hookrightarrow$ $\mathcal{L}/_{\operatorname{im}\phi}$ loc. free.

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Now denote $\Gamma(Y, \mathcal{F}) = M$, $\Gamma(Y, \mathcal{L}) = N$, $M \xrightarrow{\phi} N \to N/_{\mathrm{im}\,\phi} \to 0$

↳ we have $N/_{\mathrm{im}\,\phi}$ A-projective.

↳ $0 \to \mathrm{im}\,\phi \to N \to N/_{\mathrm{im}\,\phi} \to 0 \Rightarrow N \cong \mathrm{im}\,\phi \oplus N/_{\mathrm{im}\,\phi}$

↳ $\mathrm{im}\,\phi$ A-projective.

↳ $0 \to \ker\phi \to M \to \mathrm{im}\,\phi \to 0 \Rightarrow M \cong \ker\phi \oplus \mathrm{im}\,\phi$

↳ This provides the desired decomposition. $\square$ (Lemma 2)

$\boxed{(1) \Rightarrow (2)}$ (Corollary 2): First of all we pick $\mathcal{K}^\bullet$ given by thm.

WLOG $Y = \mathrm{Spec}\, A$, $\mathcal{K}^\bullet$ free.

Consider $d^p : K^p \to K^{p+1}$, $d^{p-1} : K^{p-1} \to \ker(d^p)$.

↳ we want $\dim(\mathrm{Im}(d^p \otimes k(y)))$ is locally constant.

Now: $\dim_{k(y)} H^p(X_y, \mathcal{F}_y) = \dim(\ker(d^p \otimes k(y))) - \dim(\mathrm{im}(d^{p-1} \otimes k(y))) =$

$= \dim(K^p \otimes k(y)) - \dim(\mathrm{Im}(d^p \otimes k(y))) - \dim(\mathrm{Im}(d^{p-1} \otimes k(y)))$

↳ everything locally constant

$$\begin{tikzcd}
K^{p-1} \arrow[r] \arrow[d, equal] & K^p \arrow[r] \arrow[d, equal] & K^{p+1} \arrow[d, equal] \\
Z^{p-1} \oplus K'_{p-1} \arrow[r] & \underbrace{B^p \oplus H_p}_{\ker d^p} \oplus K'_p \arrow[r] & B^{p+1} \oplus K'_{p+1}
\end{tikzcd}$$

iso.   iso.

Now tensor with B: $H^p(\mathcal{K}^\bullet \otimes_A B) = H^p \otimes B = H^p(\mathcal{K}^\bullet) \otimes B$,

$H^{p-1}(\mathcal{K}^\bullet \otimes_A B) = H^{p-1}(\mathcal{K}^\bullet) \otimes B$

We're still in local case, so $R^p f_* \mathcal{F} = H^p(X, \mathcal{F})^\sim$.

To use Lemma 1, we want: $\dim_{k(y)}(R^p f_* \mathcal{F} \otimes k(y)) = \dim_{k(y)}(H^p(X, \mathcal{F}) \otimes k(y))$

$\overset{!}{=} \dim(\mathcal{H}^p(\mathcal{K}^\bullet) \otimes k(y)) = \dim(\mathcal{H}^p(\mathcal{K}^\bullet \otimes k(y))) \longrightarrow$ use Lemma 1

$\boxed{\text{See saw thm}}$ (= Corollary 6):

X complete variety, ($pr_Y : X \times Y \to Y$ closed), T variety $/k$, $\mathcal{L}$ line bundle on $X \times T$, $T_1 = \{t \in T : \mathcal{L}|_{X \times \{t\}}$ - trivial on $X \times \{t\}\}$.
$\Rightarrow$ T_1 is closed in T, and $\exists$ line bundle $\mathcal{M}$ on T_1 s.t. $p^*\mathcal{M} = \mathcal{L}$, where $p : X \times T_1 \to T_1$.

$\boxed{\text{Rem}}$: If we have ~~∄~~ a trivial line bundle $\mathcal{M}'$ on complete var. X
$\Leftrightarrow h^0(X, \mathcal{M}) > 0$ and $h^0(X, \mathcal{M}^{-1}) > 0$.
$\Rightarrow$ clear as $H^0(X, \mathcal{O}_X) = k$
$\Leftarrow$: $\sigma \in H^0(X, \mathcal{M}) \setminus \{0\}$, $\delta \in H^0(X, \mathcal{M}^{-1}) \setminus \{0\}$ non-zero sections
↳ $\sigma \otimes \delta \in H^0(X, \mathcal{O}_X)$, $\sigma \otimes \delta = a \neq 0$ $\Rightarrow$ give an is $\mathcal{O}_X \simeq \mathcal{M}$.

$X \times \{t\}$ complete as well.
↳ $T_1 = \{t \in T \mid h^0(X \times \{t\}, \mathcal{L}|_{X \times \{t\}}) > 0\} \cap$
$\cap \{t \in T \mid h^0(X \times \{t\}, \mathcal{L}^{-1}|_{X \times \{t\}}) > 0\}$

<u>Recall</u> (Corollary 1): $y \longmapsto \dim_{k(y)} H^p(X_y, \mathcal{F}_y)$ upper semi cont.

↳ both parts in the intersection above are closed, so T_1 is closed.
$\Rightarrow$ endow T_1 with the reduced induced closed subscheme structure.
We want to use ~~Lemma 1 (from~~ Coroll. 2):
$\dim_{k(y)} H^0(X \times \{t\}, \mathcal{L}|_{X \times \{t\}}) = \dim_{k(y)} H^0(X \times \{t\}, \mathcal{O}_{X \times \{t\}}) = 1$
↳ $t \overset{\in T_1}{\longmapsto} \dim_{k(y)} H^0(X \times \{t\}, \mathcal{L}|_{X \times \{t\}})$ constant
$\Rightarrow R^0 p_* \mathcal{L}$ locally free (by Lemma 1) $\Rightarrow$ it is a line bundle.
Now we want to show that for $p_* \mathcal{L} =: \mathcal{M}$ we have $p^*\mathcal{M} \simeq \mathcal{L}$.
We have a natural map $p^*(p_*(\mathcal{L})) \longrightarrow \mathcal{L}$.

$$\begin{tikzcd} X \times \{t\} \arrow[r, "i"] \arrow[d, "q"'] & X \times T_1 \arrow[d, "p"] \\ \operatorname{Spec} k(t) \arrow[r, "j"] & T_1 \end{tikzcd}$$

$i^* p^*(p_* \mathcal{L}) \longrightarrow \mathcal{L}|_{X \times \{t\}}$
$\Big(\Big((pi)^*(p_* \mathcal{L}) \longrightarrow \mathcal{L}|_{X \times \{t\}}$
$(jq)^*(p_* \mathcal{L}) = q^*(j^*(p_* \mathcal{L}))$

$\simeq q^*(q_* i^* \mathcal{L}) \longrightarrow \mathcal{L}|_{X \times \{t\}}$
↳ $q^* q_* \mathcal{L}|_{X \times \{t\}} \longrightarrow \mathcal{L}|_{X \times \{t\}}$ □
↳ both sides trivial, and iso.