Abelian varieties

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Def: Let $f: X \to Y$ be a morphism of schemes, $\mathcal{F} \in QCoh(X)$. We say $\mathcal{F}$ is f-flat (also 'flat over Y') if $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module for all $x \in X$.

Note: if $Y = \operatorname{Spec} A$, the above is equivalent to: for all $\operatorname{Spec} B \subseteq X$, $\mathcal{F}|_{\operatorname{Spec} B} = \widetilde{M}$, where M is a flat A-modle.

Set-up: $f: X \to Y$ proper map of Noetherian schemes, $\mathcal{F} \in Coh(X)$ flat over Y. A is a Noetherian ring, for $y \in Y$, $X_y :=$ fiber over y, $\mathcal{F}_y := \mathcal{F} \otimes_{\mathcal{O}_Y} k(y)$.

Goal: Study the cohomology groups $H^i(X_y, \mathcal{F}_y)$ as $y \in Y$ varies. We will prove:
- $y \mapsto h^i(X_y, \mathcal{F}_y)$ is upper semi-continuous
- $y \mapsto \chi(\mathcal{F}_y)$ is a locally constant function.

*** Appendix**:

Thm 0.1: $f: X \to Y$ is a proper map, $\mathcal{F} \in Coh(X)$, then for all $i \geq 0$, $(R^i f_*)(\mathcal{F}) \in Coh(Y)$.

Lemma 0.2: Suppose $0 \to L_0 \xrightarrow{\partial'} L_1 \to L_2 \to \dots \to L_n \to 0$ be an exact sequence of A-modules such that L_i is flat for $i \geq 1$, then L_0 is flat as well.

Pf (sketch): $0 \to \operatorname{Im} \partial' \to L_2 \to \dots \to L_n \to 0$, $0 \to L_0 \to L_1 \to \operatorname{Im} \partial' \to 0$.
Take long exact sequence in $\operatorname{Tor}^A_\cdot(M, -)$.

Corollary 0.3: Suppose $0 \to L_0 \to L_1 \to \dots \to L_n \to 0 = L^\bullet$ is an exact sequence of flat modules, and let M be an A-module.
Then $L^\bullet \otimes M$ is exact.

Pf: $0 \to Z^i(L^\bullet) \to L^i \to Z^{i+1}(L^\bullet) \to 0$. Tensor with M is exact
$\rightsquigarrow$ take LES in Tor.

Def 0.4: Let $\phi: K^\bullet \to C^\bullet$ be a morphism of CoChains of A-modules. Then the mapping cone of ϕ is $L^\bullet$ where $L^i := K^i \oplus C^{i-1}$, with boundary map $\partial^i = \begin{bmatrix} \partial_K^i & 0 \\ \phi & -\partial_C^{i-1} \end{bmatrix}$

Thm 0.5: $\phi: K^\bullet \to C^\bullet$ is a quasi-isomorphism $\Leftrightarrow$ $L^\bullet$ is acyclic.

Pf: $0 \to C'^\bullet \to L^\bullet \to K^\bullet \to 0$, where $C' = C[-1]$, i.e. $C'^p = C^{p-1}$, $\partial_{C'}^p = -\partial_C^{p-1}$. Taking the LES in cohomology:

$$\cdots \to H^i(K^\bullet) \xrightarrow{\phi^i} \underset{\substack{\| \\ H^i(C^\bullet)}}{H^{i+1}(C'^\bullet)} \to H^{i+1}(L^\bullet) \to H^{i+1}(K^\bullet) \xrightarrow{\phi^{i+1}} H^{i+2}(C'^\bullet) \to \cdots$$

$\square$

Thm: Let $f, X, Y, \mathcal{F}$ be as before, with $Y = \operatorname{Spec} A$ affine. Then there exists a complex $K^\bullet$ s.t. $K^\bullet$ is a finite complex of finitely generated projective A-modules s.t. for all A-algebras B:
$H^i(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B) = H^i(K^\bullet \otimes_A B)$.

Rem: Suppose $\mathcal{U} = \{U_i\}$ is a finite affine open cover for X.
$\hookrightarrow \mathcal{U} \times_Y \operatorname{Spec} B = \{U_i \times_Y \operatorname{Spec} B\}$ is an affine open cover for $X \times_Y \operatorname{Spec} B$.
$H^i(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B)$ can be computed with the Čech cohomology:

$$C^p(\mathcal{U} \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B) = \prod_{i_0 < \dots < i_p} \Gamma(U_{i_0 \dots i_p} \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B)$$
$$= \prod_{i_0 < \dots < i_p} \Gamma(U_{i_0 \dots i_p}, \mathcal{F}) \otimes_A B = C^p(\mathcal{U}, \mathcal{F}) \otimes B.$$

So: $H^i(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B) \cong H^i(\check{C}^\bullet \otimes_A B)$
But $\check{C}$ not necessarily finitely generated!

Lemma: Suppose $(0 \to C_0 \to \dots \to C_n \to 0) = C^\bullet$ is a bounded complex of A-modules s.t. $H^i(C^\bullet)$ are finitely generated. Then there exists $K^\bullet = (0 \to K_0 \to \dots \to K_n \to 0)$ and a map $\phi: K^\bullet \to C^\bullet$ s.t.

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(i) $K^\cdot$ is a complex of fin. gen. A-modules, and for $i>0$: K^i flat

(ii) $H^i(\phi^i): H^i(K^\cdot) \xrightarrow{\sim} H^i(C^\cdot)$ is an iso $\forall i$

(iii) If each C^i is flat then so is K^0.

$\boxed{\text{Pf}}$: at the end.

$\boxed{\text{Lemma 2}}$: Let $\phi: K^\cdot \to C^\cdot$ be a quasi-iso of complexes of flat A-modules. Then for every A-algebra B we have that $\phi \otimes B : K^\cdot \otimes B \to C^\cdot \otimes B$ is a quasi isomorphism.

$\boxed{\text{Pf}}$: If $L^\cdot$ is the mapping cone for ϕ, then $L^\cdot \otimes B$ is the mapping cone for $\phi \otimes B$. L^i is flat for all i and L^i is acyclic since ϕ is a quasi-iso. $\Rightarrow L^\cdot \otimes B$ is exact

$\Rightarrow \phi \otimes B$ is a quasi-iso.

$\hookrightarrow H^i(K^\cdot \otimes B) \cong H^i(C^\cdot \otimes B) \cong H^i(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B)$.

$\boxed{\text{Corollary}}$: With $f, X, Y, \mathcal{F}$ as before, we take the following:

(i) $\varphi: Y \to \mathbb{Z}_{\geq 0}$, $y \mapsto h^i(X_y, \mathcal{F}_y) = \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$ is upper semi continuous

(ii) $y \mapsto \chi(\mathcal{F}_y)$ is locally constant.

$\boxed{\text{Pf}}$: We can assume $Y = \operatorname{Spec} A$ is affine $K^\cdot = (0 \to K_0 \to \dots \to K_n \to 0)$.
Furthermore, assume that K^i are free. We know $H^i(X_y, \mathcal{F}_y) \cong H^i(K^\cdot \otimes_A k(y))$

$$\Rightarrow h^i(K \otimes_A k(y)) = \dim_{k(y)}(\ker(d^i \otimes k(y)) - \dim_{k(y)}(\operatorname{Im}(d^{i-1} \otimes k(y)))$$

$$= \dim_{k(y)}(\underbrace{K^i \otimes k(y)}_{\text{constant}}) - \dim_{k(y)}(\operatorname{Im}(d^i \otimes k(y)))$$

$\hookrightarrow \chi(\mathcal{F}_y) = \sum (-1)^i \dim_{k(y)}(K^i \otimes k(y))$ is constant

For (i), it then suffices to prove that $y \mapsto \dim(\operatorname{Im}(d^i \otimes k(y)))$ is lower semi-cont.

$K^i \xrightarrow[A]{\partial^i} K^{i+1}$, $\Lambda^r A = (a_{ij})$

$\hookrightarrow \Lambda^r K^i \xrightarrow[\Lambda^r A]{\Lambda^r \partial^i} \Lambda^r K^{i+1}$

$\hookrightarrow \Lambda^r K^i \otimes k(y) \xrightarrow{\Lambda^r A \otimes k(y)} \Lambda^r K^{i+1} \otimes k(y)$

$$\begin{array}{ccc} \Lambda^r K^i \otimes k(y) & & \Lambda^r K^{i+1} \otimes k(y) \\ \| & & \| \\ \Lambda^r(K^i \otimes k(y)) & \xrightarrow[\Lambda^r(\partial^i \otimes k(y))]{\Lambda^r A \otimes k(y)} & \Lambda^r(K^{i+1} \otimes k(y)) \end{array}$$

If $\dim_{k(y)} \operatorname{Im}(\partial^i \otimes k(y)) < r$, then $\Lambda^r A \otimes k(y) = 0 \iff$ In $k(y)$, $a_{ij} = 0$.

$\hookrightarrow \{ y \in Y \mid \dim_{k(y)} \operatorname{Im}(\partial^i \otimes k(y)) < r \}$ is closed.

$\boxed{\text{Pf of Lemma 1}}$: We use reverse induction to construct $K^\cdot$, $\partial_K^\cdot$, ϕ^k as follows: Suppose for $i \geq m+1$, we have constructed K^i, ∂_K^i, ϕ^i:

$$\begin{tikzcd} K^m \arrow[r, dashed, "\cancel{\partial^m}"] \arrow[d, dashed, "\phi^m"'] & K^{m+1} \arrow[r, "\partial^{m+1}"] \arrow[d, "\phi^{m+1}"] & K^{m+2} \arrow[r, "\partial^{m+2}"] \arrow[d, "\phi^{m+2}"] & K^{m+3} \arrow[d, "\phi^{m+3}"] \\ \cdots C^m \arrow[r] & C^{m+1} \arrow[r] & C^{m+2} \arrow[r] & C^{m+3} \end{tikzcd}$$

Such that

(i) $\partial^{i+1} \circ \partial^i = 0$ for all $i \geq m+1$

(ii) $\partial_C^i \circ \phi^i = \phi^{i+1} \circ \partial_K^i$

(iii) $H^i(\phi) : H^i(K^\cdot) \to H^i(C^\cdot)$ is an iso for $i \geq m+2$

(iv) $\phi^{m+1}_{|\ker \partial^{m+1}} : \operatorname{Ker} \partial^{m+1} \to H^{m+1}(C)$ is surjective

denote the kernel of this by B^{m+1}

Because B^{m+1} is f.g. free A-module:

$$\begin{tikzcd} K''^m \arrow[r, "\lambda"] \arrow[dr] \arrow[ddr, bend right, "\partial''"'] & H^m(\cdot) \\ & Z^m(\cdot) \arrow[u] \\ & C^m \arrow[u] \end{tikzcd}$$

$K^m = K'^m \oplus K''^m$. Since $\phi^{m+1} \circ \partial^m \subseteq g_m \partial_C^m$ and K' is free, $\phi^{m+1} \circ \partial^m$ lifts to $\phi' : K'^m \to C^m$. $\partial^m := \partial' \oplus 0$, $\phi = \phi' \oplus \phi''$

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We have constructed $K^0 \to K^1 \to \dots \to K^n$, define $K^{-1} = 0$, and replace $K^0 = {K^0}/{\ker d^0 \cap \ker \phi^0}$.

Suppose L is the mapping cone of $\phi : K^\cdot \to C^\cdot$, then L^i is flat for $i \geq 1$, and ϕ is a quasi-iso, so $L^\cdot$ is acyclic.

$$L^\cdot = (0 \to \underset{K^0}{\underset{\shortparallel}{L^0}} \to L^1 \to \dots \to L^n \to 0), \quad L^0 \overset{!}{=} K^0 \oplus C^{-1} = K^0$$

↳ K^0 is flat.

$\boxed{\text{Remark}}$: Let $\phi : A^\cdot \to B^\cdot$ be a morphism of complexes. Then we get a sequence $A^\cdot \to B^\cdot \to \underset{\text{mapping cone}}{C(\phi)} \to A^\cdot[1]$

$$H^i(A^\cdot) \to H^i(B^\cdot) \to H^i(C(\phi)) \to H^{i+1}(A^\cdot) \to H^{i+1}(B^\cdot)$$

$\boxed{\text{Remark}}$ (Flatness is necessary):

$$\begin{array}{c} s \in X \times T \\ \downarrow \mathrm{pr}_T \\ T \end{array}$$

X complete variety, T quasi-proj. / $k = \bar{k}$
$s \in$ closed points
M_s ideal sheaf of s.

↳ $H^0(X \times \{t\}, M_s|_{X \times \{t\}})$

$t \neq f(s)$: $\cong k$

$t = f(s)$: $\cong 0$