Abelian varieties

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Reminder

① X complex torus

② $V \overset{\cong_{v.s.} \mathbb{C}^n}{=} T_e X \xrightarrow[\pi]{} X$ homomorphism of Lie groups

③ $\Lambda = \ker(\pi) \underset{grp}{\cong} \mathbb{Z}^{2n}$

④ $\operatorname{Pic} X \longleftrightarrow \left\{ (H, \alpha) \;\middle|\; \begin{array}{l} ⓐ \text{ Hermitian on } V \\ ⓑ \operatorname{im} H(\Lambda \times \Lambda) \subseteq \mathbb{Z} \\ ⓒ\ \alpha : \Lambda \longrightarrow \mathbb{C}_1^* = \{ z \in \mathbb{C} \mid |z| = 1 \} \\ ⓓ\ \alpha(g+g') = \alpha(g)\alpha(g') e^{i\pi E(g,g')} \text{ for } E = \operatorname{Im} H \end{array} \right\}$

line bundle $L(H,\alpha)$ associated to (H,α):

$$L(H,\alpha) = \mathbb{C} \times V \Big/ (t,z) \mapsto (e_g(z) \cdot t,\ z+g) \quad , \quad e_g(z) = \alpha(g) e^{\pi H(z,g) + \frac{1}{2}\pi H(g,g)}$$

⑤ $H^0(X, L(H,\alpha)) \longleftrightarrow \theta : V \longrightarrow \mathbb{C}$ s.t. $\theta(z+g) = e_g(z) \cdot \theta(z)$.

⑥ ~~Theory~~ Last time: Recall $L(H,\alpha)$ ample ($\overset{def}{\Longleftrightarrow} \exists N > 0$ s.t. L^N induces $X \overset{\phi_L}{\underset{closed}{\hookrightarrow}} \mathbb{P}^n$
$x \longmapsto [\theta_0(y) : \ldots : \theta_n(y)]$
$y \in \pi^{-1}(x)$, $\{\theta_i\}_i$ basis)

$\Rightarrow H$ pos. def.

⑦ Today:

Thm 1: H pos. def $\Rightarrow \dim_{\mathbb{C}} H^0(X, L(H,\alpha)) = \sqrt{\det E} \Rightarrow$ Cor: H pos. def. $\Rightarrow$ $\phi_{L(H,\alpha)}$ exists as rat. map.

Rem: There will be an algebraic version of this.

Thm 2 (Lefschetz): $L(H,\alpha)$ ample $\Leftrightarrow H$ pos. def.

Pf of Thm 1:

Recall: Fourier ~~Koeff~~ series: $f : \mathbb{C} \xrightarrow{hol.} \mathbb{C}$, $f(z+1) = f(z)$, $c_n = \int f(z) \cdot e^{-2\pi i z n}$,
$\hookrightarrow f(z) = \sum c_n e^{2\pi i n z}$

higher dimensions: $1 \rightsquigarrow g \in \Lambda' \underset{grp}{\cong} \mathbb{Z}^{\oplus n}$

$n \rightsquigarrow x \in \widehat{\Lambda'} = \operatorname{Hom}(\Lambda', \mathbb{Z})$.

last time: We found $\Lambda' \subseteq \Lambda$ s.t. [ⓐ $E(\Lambda' \times \Lambda') \equiv 0$] [ⓑ $\Lambda' \underset{grp}{\cong} \mathbb{Z}^n$]
[ⓒ $W \overset{def}{=} \Lambda' \otimes_{\mathbb{Z}} \mathbb{R} \Rightarrow W \oplus iW = V \Rightarrow \Lambda' \otimes_{\mathbb{Z}} \mathbb{C} = V$]

[ⓓ $H|_{\Lambda'\times\Lambda'}$ real symm. $\rightsquigarrow$ $B: V\times V \longrightarrow \mathbb{C}$ unique complex symm. extension]

[ⓔ $H - B|_{W\times W} \equiv 0$

$\Downarrow$ $\mathbb{C}$-linearity in 1st variable

$0 = (H-B)(z,w) \quad (z\in V,\ w\in W)$]

[ⓕ By ⓐ, $\alpha: \Lambda' \longrightarrow \mathbb{C}_1^*$ is a group homomorphism

$\hookrightarrow$ $\alpha(g) = e^{2\pi i \lambda(g)}$, $\quad \lambda \in \underset{(\cong \operatorname{Hom}_{\mathbb{R}}(W,\mathbb{R}))}{\operatorname{Hom}_{\mathbb{C}}(V,\mathbb{C})}$. (real valued on W)]

<u>Key</u>: Turn ϑ into Λ'-periodic using B and α.

Define $\vartheta^*(z) = \vartheta(z)\, e^{-\frac{1}{2}\pi B(z,z)}$. As $\vartheta(z+g) = \alpha(g)\cdot e^{\pi H(z,g) + \frac{1}{2}H(g,g)}\,\vartheta(z)$,

we obtain $\vartheta^*(z+g) = \alpha(g)\cdot e^{\pi(H-B)(z,g) + \frac{1}{2}(H-B)(g,g)}\cdot \vartheta^*(z)$

$\left(e^{-\frac{1}{2}\pi B(z+g,\,z+g)} = e^{-\frac{1}{2}\pi B(z,z) - \pi B(z,g) - \frac{1}{2}\pi B(g,g)} \right)$

$\rightsquigarrow$ if $g\in\Lambda'$ then $\pi(H-B)(z,g) + \frac{1}{2}(H-B)(g,g) = 0$, $\alpha(g) = e^{2\pi i \lambda(g)}$.

$\rightsquigarrow$ $e^{-2\pi i\lambda(z)}\cdot\vartheta^*(z)$ is Λ'-periodic

$\rightsquigarrow$ $e^{-2\pi i\lambda(z)}\,\vartheta^*(z) = \sum\limits_{\chi\in\widehat{\Lambda'}} c_\chi\, e^{2\pi i \chi(z)}$ $\Rightarrow$ $\vartheta^*(z) = \sum\limits_{\chi\in\widehat{\Lambda'}} c_\chi\, e^{2\pi i(\chi(z)+\lambda(z))}$ $\quad (**)$

⑤ gives for c_χ: (for $g \in \underset{\subseteq V}{\Lambda}$ define $\hat{g} \in \widehat{\Lambda'} \in \operatorname{Hom}_{\mathbb{Z}}(\Lambda',\mathbb{Z}) \subseteq \underset{=\operatorname{Hom}_{\mathbb{C}}(V,\mathbb{C})}{\operatorname{Hom}_{\mathbb{R}}(W,\mathbb{R})}$

by $\hat{g}(g') = E(g',g)$, where $g'\in\Lambda'$.)

$c_\chi = \alpha(g)\cdot e^{i\pi\hat{g}(g) - 2\pi i(\chi(g)+\lambda(g))}\cdot c_{\chi-\hat{g}} \qquad (\Delta)$

This equation characterizes ϑ as soon as $(**)$ is convergent.

<u>matrix of E</u>:

Λ' $\quad$ Λ'' a complement of Λ'

$$\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{array}{ccc} \Lambda & \longrightarrow & \widehat{\Lambda'} \\ \Psi & & \Psi \\ g & \longmapsto & \hat{g} = (pr_{\Lambda''}(g)\cdot A\cdot(-)) \end{array}$$

$\hookrightarrow$ map given by A after appropriate identifications.

$\cancel{|\det E| = |\det A|^2}$

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Notice $|\det E| = |\det A|^2 \Rightarrow \sqrt{|\det E|} = \det A = \overbrace{|\operatorname{coker} A|}^{M=}$.

Need: $\left\{ \begin{matrix} \text{cosets} \\ \text{of } M \end{matrix} \right\} \longleftrightarrow \left\{ \vartheta \overset{\text{basis of}}{\text{functions}} \right\}$.

Now (Δ) gives that c_x on any representative of any coset of M determines the value of c_x on the whole coset. (+ convergence ~ book) $\square$

$\boxed{\text{Pf of Thm 2}}$: Thm 1 $\Rightarrow \exists \vartheta \neq 0$. So take $\vartheta \in H^0(L(H,\alpha)) \setminus \{0\}$

ϑ satisfies ⑤. Define $\vartheta^{a,b}(z) = \vartheta(z-a)\vartheta(z-b)\vartheta(z+a+b)$, $a,b \in V$.

Claim: this is a ϑ-function for $(3H, \alpha_4^3)$

Pf of Claim: $\vartheta(z-a+g)\vartheta(z-b+g)\vartheta(z+a+b+g) =$

$$= \underbrace{\alpha_4(g)^3}_{=\alpha_{3H}(g)} \cdot e^{\pi \underbrace{H(3z,g)}_{=3H(z,g)} + \underbrace{\frac{3}{2}\pi H(g,g)}_{=\frac{1}{2}\pi(3H)(g,g)}} \vartheta(z-a)\vartheta(z-b)\vartheta(z+a+b) \quad \square \text{ (Pf of Claim)}$$

Note: ~~$\vartheta(a) \neq 0, \vartheta(b) \neq 0, \vartheta(a+b) \neq 0$~~

if $\vartheta(z+a) \neq 0$, $\vartheta(z+b) \neq 0$, $\vartheta(z+a+b) \neq 0$, then $\vartheta^{a,b}(z) \neq 0$

As the non-vanishing locus of ϑ is open dense, we have:

Claim: $\forall z \ \exists a,b : \vartheta^{a,b}(z) \neq 0$

$\hookrightarrow \underset{=(L(H,\alpha)^{\otimes 3})}{L^3}$ is basepoint free. $\Rightarrow L(H,\alpha)^{\otimes 3}$ defines a morphism $\xi : X \longrightarrow \mathbb{P}^n$.

(Replace $L(H,\alpha)$ by $L(H,\alpha)^{\otimes 3}$.)

Need: ξ injective. → up to replacing L by $L^{\otimes 3}$

Suppose by contradiction: $\xi(\pi(z_1)) = \xi(\pi(z_2))$ with $\pi(z_1) \neq \pi(z_2)$ $\Rightarrow \exists \gamma \in \mathbb{C}^*$ s.t. $\vartheta(z_2) = \gamma \cdot \vartheta(z_1)$.

Let $\sigma = z_2 - z_1 \notin \Lambda$. Point of the proof: every ϑ function for Λ is a ϑ function for $\Lambda + \sigma \cdot \mathbb{Z}$ ($[\Lambda + \sigma\mathbb{Z} : \Lambda] < +\infty$).

But $\det(\Lambda + \sigma \cdot \mathbb{Z}) < \det(\Lambda) \Rightarrow$ contradiction /w Thm 1.

Idea: $\vartheta(z_2-a)\,\vartheta(z_2-b)\,\vartheta(z_2+a+b) = \vartheta(z_1-a)\,\vartheta(z_1-b)\,\vartheta(z_1+a+b)\cdot\gamma$

$\left(\log(\dots)\right)'^{\text{ w.r.t. } a} =: \omega \quad \Rightarrow \quad -\omega(z_2-a)+\omega(z_2+a+b) = -\omega(z_1-a)+\omega(z_1+a+b)$

logarithmic derivative w.r.t. a

$\Rightarrow \omega(z_2+z)-\omega(z_2-z)$ translation invariant

$\Rightarrow \omega$ constant

$\rightsquigarrow \vartheta(z+z_2) = e^{\overset{\text{linear}}{l(z)}}\,\vartheta(z+z_1)$

$\rightsquigarrow \vartheta(z+b) = \underset{\mathbb{C}^\times}{A}\, e^{l(z)}\cdot\vartheta(z)$

$\rightsquigarrow$ looks similar to ⑤, and after some more work it gives a ϑ-function for $\Lambda+b\mathbb{Z}$.