Abelian varieties

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$\boxed{\text{Goal}}$: Appel - Humbert : $X = V/\Lambda$, $\dim_{\mathbb{C}} X = n$, $\pi: V \to X$ loc. biholo, $H = \Gamma(\pi^* \mathcal{O}_X) = \Gamma(\mathcal{O}_V)$, $H^\times = \Gamma(\pi^* \mathcal{O}_X^\times) = \Gamma(\mathcal{O}_V^\times)$

$H^{0,2}$

$\boxed{\text{Rem}}$: $0 \to \mathrm{Pic}^0 X \to \mathrm{Pic}\, X \to \mathrm{Ker}(H^2(X,\mathbb{Z}) \to H^2(X, \mathcal{O}_X))$

$\hookrightarrow H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$ as $\mathbb{C}$-v.sp.

conjugate

$$\begin{array}{ccccccc}
0 \to & \mathrm{coker}(H^1(X,\mathbb{Z}) \to H^1(X,\mathbb{C})) & \to & H^1(X,\mathcal{O}_X) & & & \\
& \uparrow \phi \ \cong & & \phi \uparrow \ \downarrow \phi^{-1} & & \uparrow \phi \ \cong & \\
0 \to & \mathrm{coker}(H^1(\Lambda,\mathbb{Z}) \to H^1(\Lambda,H)) & \to & H^1(\Lambda, H^\times) & \xrightarrow{\delta} & \ker(H^2(\Lambda,\mathbb{Z}) \to H^2(\Lambda, H^\times)) & \to 0
\end{array}$$

$\phi^{-1}(L)$: $\pi^* L \cong \mathbb{C} \times V$ $\qquad L = e \longmapsto$? explicit descr. ? $= \cancel{\dots}$

$\Lambda \circlearrowright$ $\quad g \in \Lambda \mapsto \alpha_g \in \mathrm{Aut}_g(\mathbb{C} \times V)$

$$\begin{array}{ccc}
\mathbb{C} \times V & \xrightarrow{\alpha_g} & \mathbb{C} \times V \\
\downarrow & \circlearrowright & \downarrow \\
V & \longrightarrow & V
\end{array}$$

$\alpha_g(t, z) = (\underbrace{e_g(z)}_{\in \mathbb{C}^\times} \cdot t, z + g)$

$\hookrightarrow e_g \in H^\times \rightsquigarrow e: \mathrm{Hom}_{set}(\Lambda, H^\times)$ (check that this has 0 boundary etc)

$\rightsquigarrow \phi^{-1}(L) = \{e\} \in H^1(\Lambda, H^\times)$.

Understanding: $H^*(\Lambda, \mathbb{Z}) \xrightarrow{\delta} H^*(X, \mathbb{Z}) \cong \Lambda^* H^1(X,\mathbb{Z})$

Take $[h] \in H^*(\Lambda, \mathbb{Z})$: $h \in \mathrm{Hom}_{set}(\Lambda, \mathbb{Z})$, $0 \overset{?}{=} (\delta h)_{g,\tilde{g}} = \cancel{g} h(\tilde{g}) - h(g+\tilde{g}) + h(g)$ action is trivial on $\mathbb{Z}$

$\hookrightarrow h$ is a co-chain $\Leftrightarrow h \in \mathrm{Hom}_{grp}(\Lambda, \mathbb{Z})$

$h_g = (\delta \tilde{h})_g = g\tilde{h} - \tilde{h} = \tilde{h} - \tilde{h} = 0$. $\Rightarrow$ no non-trivial co-boundaries.

$\mathbb{Z} = \mathrm{Hom}_{set}(\Lambda^0, \mathbb{Z})$

$\Rightarrow$ canonical $H^1(\Lambda, \mathbb{Z}) = \mathrm{Hom}_{grp}(\Lambda, \mathbb{Z})$.

$X \underset{diff}{\cong} (S^1)^{2n}$. Note that $H^*(S^1, \mathbb{Z}) \underset{\text{as alg's}}{=} \Lambda^* H^1(S^1, \mathbb{Z})$ $\quad \mathbb{Z}[x]/(x^2)$

$\Longrightarrow$ Künneth formula comp. w/ cup product. $\quad H^*(X,\mathbb{Z}) = \bigotimes^{2n} \Lambda^* H^1(S^1, \mathbb{Z}) = \Lambda^* H^1(X, \mathbb{Z})$

$H^1(X,\mathbb{Z}) \underset{\text{Künneth}}{\cong} \bigoplus^{2n} H^1(S^1,\mathbb{Z})$

So:
$$\begin{array}{ccc}
H^*(\Lambda, \mathbb{Z}) & \xrightarrow[\cong]{\delta} & H^*(X,\mathbb{Z}) \\
\uparrow \cong & & \| \wr \\
\Lambda^* H^1(\Lambda, \mathbb{Z}) & \xrightarrow[\Lambda^*(\delta)]{\cong} & \Lambda^* H^1(X,\mathbb{Z})
\end{array}$$

cup product

$\cong \Lambda^* \mathrm{Hom}_{grp}(\Lambda, \mathbb{Z}) \underset{n=2}{\overset{\text{specialize}}{\Longrightarrow}} H^2(\Lambda, \mathbb{Z}) \cong \Lambda^2 \mathrm{Hom}_{grp}(\underset{\mathbb{Z}^{2n}}{\Lambda}, \mathbb{Z})$

$\cong \mathbb{Z}^{2n}$ non. can.

$= \{$alternating bilinear forms on $\Lambda\}$

Cup product for group cohomology: $\phi, \psi \in \mathrm{Hom}_{gp}(\Lambda, \mathbb{Z}) \rightsquigarrow \phi \cup \psi \in H^2(\Lambda, \mathbb{Z})$

action on $\mathbb{Z}$ trivial — multiplication on $\mathbb{Z}$

given by $(\phi \cup \psi)_{g,g'} = \phi(g) \cdot \psi(g')$

$\rightsquigarrow$ wedge product: $\phi(g)\psi(g') - \phi(g')\psi(g)$

So: $H^2(X, \Lambda) \cong \{\text{alternating bilinear } \Lambda \to \mathbb{Z}\}$

$$\mathrm{Hom}_{set}(\Lambda^2, \mathbb{Z}) \ni F \longmapsto \Big((g, \tilde{g}) \mapsto \underbrace{F(g,\tilde{g}) - F(\tilde{g}, g)}_{AF(-,-)} \Big)$$

$f \mapsto e^{2\pi i f}$

$0 \to \mathbb{Z} \to \mathcal{H} \to \mathcal{H}^\times \to 0$

$$\text{For } H^1(\Lambda, \mathcal{H}^*) \xrightarrow{\delta} \ker(H^2(\Lambda, \mathbb{Z}) \to H^2(\Lambda, \mathcal{H}^\times))$$

$$L = e \longmapsto \underbrace{f_{v_2}(z+v_1) - f_{v_1+v_2}(z) + f_{v_1}(z)}_{= F(v_1, v_2)} \in \mathbb{Z}$$

doesn't depend on $\mathbb{Z}$

$$\leftrightarrow AF(v_1, v_2) = F(v_1,v_2) - F(v_2, v_1) = f_{v_2}(z+v_1) + f_{v_1}(z) - f_{v_1}(z+v_2) - f_{v_2}(z)$$

of AF

$\boxed{\text{Cor}}$: $\exists!\ \mathbb{R}$-lin. ext. $E: V \to \mathbb{R}$ " $\Rightarrow E(x,y) = E(ix, iy)$ ($\rightsquigarrow$ Hodge decomposition of H^2 is used)

algebraic lemma: $\left\{ \begin{array}{l} \text{anti-symm. real bilinear} \\ \text{forms on } V \text{ satisfying} \end{array} \right\} \longleftrightarrow \{\text{Hermitian forms}\}$

$$E \longmapsto E(ix, y) + i E(x,y)$$
$$\mathrm{Im}\, H \longleftarrow H$$

$\Rightarrow$ we obtain a map $\ker(H^2(\Lambda, \mathbb{Z}) \to H^2(\Lambda, \mathcal{H}^*)) \xrightarrow[\text{actually iso.}]{} \left\{ \begin{array}{l} H: \text{ Hermitian form on } V \\ \text{satisfying } H(\Lambda \times \Lambda) \subseteq \mathbb{Z} \end{array} \right\}$

fix: $H: V \times V \to \mathbb{C}$ Hermitian want: f_v that gives H

know: $\mathrm{Im}\, H = f_{v_2}(z+v_1) + f_{v_1}(z) - f_{v_1}(z+v_2) - f_{v_2}(z)$.

Note: $f_v(z) = \frac{1}{2i} H(z, v) + \beta_v$ works.
$\beta_v \in \mathbb{C}$

Because: $$\frac{1}{2i}\Big(\underbrace{H(z+v_1, v_2)}_{= H(z,v_2) + H(v_1,v_2)} + H(z, v_1) - \underbrace{H(z+v_2, v_1)}_{= H(z,v_1) + H(v_2,v_1)} - H(z, v_2) \Big) = \frac{1}{2i}\big(H(v_1,v_2) - H(v_2,v_1)\big)$$
$$= \mathrm{Im}\, H \in \mathbb{Z}$$

(β cancels as well)

One more condition: $f_{v_2}(z+v_1) - f_{v_1+v_2}(z) + f_{v_1}(z) \in \mathbb{Z}$

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by plugging in, equivalent to: $\frac{1}{2} H(v_1,v_2) + i(\beta_{v_1} + \beta_{v_2} - \beta_{v_1+v_2}) \in \mathbb{Z}$

Introduce: $\gamma_v = i\beta_v - \frac{1}{4} H(v,v)$

↳ equivalent to

$$\gamma_{v_1} + \gamma_{v_2} - \gamma_{v_1+v_2} + \frac{1}{2} i \overbrace{E(v_1,v_2)}^{\mathrm{Im}\, H(v_1,v_2)} \in i\mathbb{Z} \quad \rightarrow (*)$$

↳ $\mathrm{Re}\,\gamma : \Lambda \to (\mathbb{R},+)$ is a group homomorphism

$$\begin{tikzcd} \Lambda \arrow[r, "\mathrm{Re}\,\gamma"] \arrow[d, phantom, "\cap"] & (\mathbb{R},+) \\ V \arrow[ur, "\exists!"'] & \end{tikzcd}$$

$\Rightarrow \exists!\, L \in \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ s.t. $\mathrm{Re}\, L = \mathrm{Re}\,\gamma$

↳ replace $\gamma.$ by $\gamma - L$ (doesn't change $f_{v_2}(z+v_1) - f_{v_1+v_2}(z) + f_{v_1}(z)$ etc.)

⇝ wma $\mathrm{Re}\,\gamma = 0 \Rightarrow \alpha(v) \overset{\mathrm{def}}{:=} e^{2\pi i \gamma_v} \rightsquigarrow |\alpha(v)| = 1.$

translating (*) for α: $\frac{\alpha(v_1+v_2)}{\alpha(v_1)\alpha(v_2)} = e^{\pi i \overbrace{E(v_1,v_2)}^{\in \mathbb{Z}}} \in \{\pm 1\}$ (**)

Note: there is always a solution: ~~[illegible]~~. Later, next time

$$e_v(z) = e^{2\pi i f_v(z)} = e^{2\pi i \left(\frac{1}{2i} H(z,v) + \beta_v\right)} = e^{\pi H(z,v) + 2\pi \frac{1}{4} H(v,v)} \cdot \alpha(v)$$

$\boxed{\text{Homework}}$ (especially for Jeff): Is there a Hermitian metric on $\mathbb{C} \times V$ invariant under the Λ-action such that the curvature is H?

$\boxed{\text{Homework}}$: [① this is a cocycle] [② α exists if H given]

↳

$$H^1(\Lambda, H^*) \to \left\{ \begin{array}{l} H: \text{Hermitian form on } V \\ \text{satisfying } H(\Lambda \times \Lambda) \subseteq \mathbb{Z} \end{array} \right\}$$

$$H^1(\Lambda, H^*) \twoheadrightarrow \ker\left(H^2(\Lambda,\mathbb{Z}) \to H^2(\Lambda, H^*)\right)$$

$$H^1(\Lambda, H^*) \to \ker\left(H^2(X,\mathbb{Z}) \to H^2(X, \mathcal{O}_X)\right)$$

all with same kernel
↳ all isomorphic.

$\Rightarrow$ we can describe $\left\{ (\alpha, H) \;\middle|\; \begin{array}{l} \cdot\ H \text{ herm.} \\ \cdot\ H(\Lambda \times \Lambda) \subseteq \mathbb{Z} \\ \cdot\ \alpha : \Lambda \to \{ z \in \mathbb{C} \mid |z| = 1 \} \\ \quad \text{satisfying } (**) \end{array} \right\} \twoheadrightarrow L \in H^1(\Lambda, H^\times)$

~~Step~~ ↳ Show that this is injective.

↳

$$\begin{tikzcd}
0 \arrow[r] & \left\{ \alpha \;\middle|\; \begin{array}{c} \alpha : \Lambda \to \{\dots\} \\ \text{group homom.} \end{array} \right\} \arrow[r] \arrow[d] & \{ (\alpha, H) \mid \dots \} \arrow[r] \arrow[d] & \left\{ H \;\middle|\; \begin{array}{c} H \text{ Hermitian form} \\ \text{on } V \text{ satisfying } H(\Lambda \times \Lambda) \subseteq \mathbb{Z} \end{array} \right\} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{coker}\left(H^1(\Lambda, \mathbb{Z}) \to H^1(\Lambda, H) \right) \arrow[r] & H^1(\Lambda, H^\times) \arrow[r] & \ker\left(H^2(\Lambda, \mathbb{Z}) \to H^2(\Lambda, H^\times) \right) \arrow[r] & 0
\end{tikzcd}$$