Abelian varieties

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Linus Rösler

$\boxed{\text{Reminder}}$: M is a G-module $\Rightarrow H^i(G,M) = \{$ derived functor of $\underset{\substack{\uparrow \\ G\text{-module} \\ (\mathbb{Z}[G]\text{-module})}}{M} \longmapsto \underset{\substack{\uparrow \\ \text{ab. gp.}}}{M^G}$

$= h^i(C^\bullet(G,M))$

$C^j(G,M) \overset{\text{def}}{=} \mathrm{Hom}_{\mathrm{Set}}(G^j, M) \ni f$

$$\delta^j(f)(g_0,\dots,g_j) = g_0 \cdot f(g_1,\dots,g_j) + \sum_{i=1}^{j-1} (-1)^{i+1} f(g_0,\dots,g_i \cdot g_{i+1},\dots,g_j) + \underbrace{\cancel{}}_{+(-1)^{j+1} f(g_0,\dots,g_{j-1})}$$

$\boxed{\text{Setup}}$: $\underset{\substack{\mathbb{C}^n \\ \wr\|}}{V} = T_0 X \xrightarrow{\pi} \overset{\text{cpx torus}}{X}$

- π surj. morph. of Lie-gps
- $\underset{\Lambda \overset{\|}{} \underset{gp}{\cong} \mathbb{Z}^{2n}}{\ker \pi} \subseteq V$ discrete subgroup

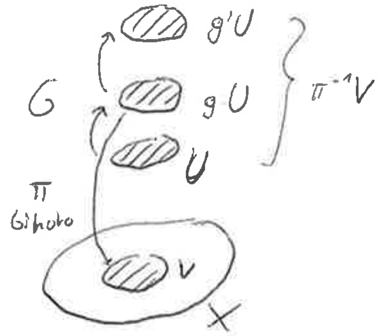
Key to Appel-Humbert: $H^i(\overset{\Lambda}{G}, \Gamma(\pi^* \mathcal{F})) \underset{\cong}{\xrightarrow{\phi^i}} H^i(\mathcal{F})$ $(*)$

Always: $H^i(\pi^*\mathcal{F}) = 0 \ (i>0)$.

$$\begin{array}{cccc} \mathcal{F} = & \mathbb{Z}_X, & \mathcal{O}_X, & \mathcal{O}_X^* \\ \pi^* & \downarrow & \downarrow & \downarrow \\ & \mathbb{Z}_V & \mathcal{O}_V & \mathcal{O}_V^* \end{array}$$

you need it only for $\cong$.

$\overset{=\Lambda}{G}$-action on $\pi^*\mathcal{F}$:



π locally biholo $\Rightarrow \pi^{-1}\mathcal{F}|_{gU} = \pi^*\mathcal{F}|_{gU}$

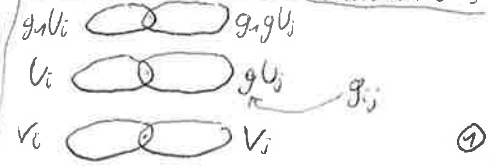
also $\pi^*\mathcal{F}|_{gU} \overset{\text{canonically}}{\cong} \mathcal{F}|_V$

$\pi^*\mathcal{F}|_{gU} \underset{\text{group action}}{\cong} \pi^*\mathcal{F}|_U$

Want to use Čech here $(*)$: $\{V_i\} \underset{\text{open cover}}{\subseteq} X$ s.t. [① $V_{i_0,\dots,i_j} \overset{\text{def}}{=} V_{i_0} \cap \dots \cap V_{i_j}$ contractible]

(Homework: Such an open cover exist) [② $\pi^{-1}V_i = \coprod_{g\in\Lambda} g\overset{\leftarrow \text{fixed}}{U_i}$ s.t. $\pi: gU_i \xrightarrow[\text{biholo}]{} V_i$ (automatic]

[③ $\forall i,j$ $\exists$ at most one $g \in \Lambda$ s.t. $U_i \cap gU_j \neq \emptyset$]
($\sim$ define $g_{ij} = g$)



Čech complex: $\check{C}^j(\mathcal{F}) = \prod_{i_0<\dots<i_j} \mathcal{F}(V_{i_0,\dots,i_j}) \ni f$

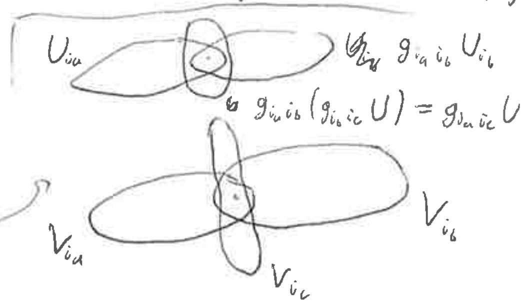
$$\check{\delta}^j(f)_{i_0,\dots,i_{j+1}} = \sum_{s=0}^{j+1} (-1)^s f_{i_0,\dots,\hat{i_s},\dots,i_{j+1}}\Big|_{V_{i_0,\dots,i_{j+1}}}$$

Definition of ϕ^j: $\phi^j : C^j(\Lambda, \Gamma(\pi^*\mathcal{F})) \xrightarrow{\ni f:\Lambda^j \to \Gamma(\pi^*\mathcal{F})} \check{C}^j(\mathcal{F})$ s.t. $\delta\phi = \phi\delta$

$\hookrightarrow (\phi^j f)_{i_0,\dots,i_j} = f(g_{i_0 i_1}, g_{i_1 i_2}, \dots, g_{i_{j-1} i_j})\big|_{U_{i_0} \cap g_{i_0 i_1} U_{i_1} \cap \dots \cap g_{i_0 i_j} U_{i_j}} \ (\xrightarrow[\pi]{\sim} V_{i_0,\dots,i_j})$

(Assume $V_{i_0,\dots,i_j} \neq \emptyset \Rightarrow g_{i_a i_b}$ exist $\forall\, 0 \leq a,b \leq j$)

$g_{i_a i_b} \cdot g_{i_b i_c} = g_{i_a i_c}$:



Why is ϕ a co-chain homom.: $(U := U_{i_0} \cap g_{i_0 i_1} U_{i_1} \cap \dots \cap g_{i_0 i_{j+1}} U_{i_{j+1}})$

$$(\hat{\delta}^j \phi^j f)_{i_0,\dots,i_{j+1}} = \underbrace{f(g_{i_1 i_2}, \dots, g_{i_j i_{j+1}})\big|_{g_{i_0 i_1}^{-1} U}}_{\text{"remove } i_0\text{"-term}} + \underbrace{\sum_{s=0}^{j-1} (-1)^{s+1} f(g_{i_0 i_1}, \dots, \overset{g_{i_s i_{s+1}} g_{i_{s+1} i_{s+2}}}{\overset{\shortparallel}{g_{i_s i_{s+2}}}}, \dots, g_{i_j i_{j+1}})\big|_U}_{\text{"remove } i_{s+1}\text{"-terms for } 0 \leq s < j}$$

$$= (g_{i_0 i_1} f(g_{i_1 i_2}, \dots, g_{i_j i_{j+1}}))\big|_U + (-1)^{j+1} f(g_{i_0 i_1}, \dots, g_{i_{j-1} i_{j+1}})\big|_U$$

$\hookrightarrow$ Compare with δ^j of group cohomology ✓

$\hookrightarrow$ we obtain $\phi^j : H^j(\Lambda, \Gamma(\pi^*\mathcal{F})) \longrightarrow H^j(X, \mathcal{F})$.

We want to show that it is an iso.

$\hookrightarrow$ induction on j: $\boxed{j=0}$: $H^0(\Lambda, \Gamma(\pi^*\mathcal{F})) = \Gamma(\pi^*\mathcal{F})^\Lambda$

$H^0(\mathcal{F}) = \Gamma(\mathcal{F}) \overset{\phi^0}{\hookleftarrow}$ is the obvious map (Homework)

$\boxed{j>0}$: $0 \longrightarrow \mathcal{F} \underset{\text{inj.}}{\longrightarrow} \mathcal{F}' \longrightarrow \underset{\text{coker}}{\mathcal{F}''} \longrightarrow 0 \quad (*)$

We assumed $H^i(\pi^*\mathcal{F}) = 0$ for $i > 0$ $\rightsquigarrow$ should have the same for $\mathcal{F}'$ & $\mathcal{F}''$

$\mathcal{F}'$ inj $\Rightarrow$ $\mathcal{F}'$ flasque $\overset{\pi \text{ local bihole}}{\Rightarrow}$ $\pi^*\mathcal{F}'$ flasque $\Rightarrow H^i(\pi^*\mathcal{F}') = 0$ for $i > 0$

$\underset{\text{by } (*)}{\Rightarrow}$ same for $\mathcal{F}''$

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<u>Also:</u> $0 \to \Gamma(\pi^*\mathcal{F}) \to \Gamma(\pi^*\mathcal{F}') \to \Gamma(\pi^*\mathcal{F}'') \to 0 \quad (**) \quad$ exact

group coho to $(**)$

$$\begin{tikzcd}
\underset{\text{by Lemma}}{0 =} H^{j-1}(\Lambda, \Gamma(\pi^*\mathcal{F}')) \arrow[r] \arrow[d, "\phi^{j-1}_{\mathcal{F}'}"] & H^{j-1}(\Lambda, \Gamma(\pi^*\mathcal{F}'')) \arrow[r] \arrow[d, "\phi", "\text{by ind.} \cong"'] & H^j(\Lambda, \Gamma(\pi^*\mathcal{F})) \arrow[r] \arrow[d, "\phi"] & H^j(\Lambda, \Gamma(\pi^*\mathcal{F}')) \underset{\text{by Lemma}}{= 0} \arrow[d, "\phi"] \\
H^{j-1}(\mathcal{F}') = 0 \arrow[r] & H^{j-1}(\mathcal{F}'') \arrow[r] & H^j(\mathcal{F}) \arrow[r] & H^j(\mathcal{F}') = 0
\end{tikzcd}$$

<u>Homework:</u> commutes

$\boxed{\text{Lemma}}$: $\Gamma(\pi^*\mathcal{F}')$ is an inj. $\overset{=\Lambda}{G}$-module

$\boxed{\text{Pf}}$: M is a G-module $\rightsquigarrow$ M_V is a G-constant sheaf

$\rightsquigarrow (\pi_* M_V) \circlearrowleft G$

π local homeo $\Rightarrow \pi^*$ exact

<u>Key:</u> $\mathrm{Hom}_G(M, \Gamma(\pi^*\mathcal{F}')) \underset{\substack{\uparrow \\ \text{all sections of} \\ M_V \text{ are global}}}{\cong} \mathrm{Hom}_G(M_V, \pi^*\mathcal{F}') \cong \mathrm{Hom}((\pi_* M_V)^G, \mathcal{F}')$

↳ Apply this to $M_1 \subseteq M_2$ of G-modules $\rightsquigarrow$ surj. on the right $\rightsquigarrow$ surj. on the left

$\underset{\mathcal{F}' \text{ inj.}}{\Rightarrow}$

$$\begin{tikzcd}
\mathrm{Hom}((\pi_* M_{2,V})^\Lambda, \mathcal{F}') \arrow[r, two heads] \arrow[d, phantom, "\wr\|"] & \mathrm{Hom}((\pi_* M_{1,V})^\Lambda, \mathcal{F}') \arrow[d, phantom, "\wr\|"] \\
\mathrm{Hom}_G(M_2, \Gamma(\pi^*\mathcal{F}')) \arrow[r, two heads] & \mathrm{Hom}_G(M_1, \Gamma(\pi^*\mathcal{F}'))
\end{tikzcd}$$

$\Rightarrow \Gamma(\pi^*\mathcal{F}')$ inj. $\square$

$\boxed{\text{Line bundles}}$:

L line bdl on X

globals $\updownarrow$ local sections

$\mathcal{L}$ inv. sheaf

$\pi^* L \xleftarrow[\cong]{\xi} \mathbb{C} \times V \cong \mathcal{O}_V$ ($\mathrm{Pic}(V) = 1$)

Λ acts linearly $\longleftrightarrow$ Λ acts linearly

$\forall g \in \Lambda : \exists \alpha_g : \mathbb{C} \times V \to \mathbb{C} \times V$

compatible with Λ-action on V

$$\begin{tikzcd}
\mathbb{C} \times V \arrow[r, "\alpha_g"] \arrow[d, "pr_2"'] & \mathbb{C} \times V \arrow[d, "pr_2"] \\
V \arrow[r, "g"'] & V
\end{tikzcd}$$

& α_g linear in the first coordinate

↳ $\alpha_g(-, z) : \mathbb{C} \times \{z\} \xrightarrow{\cong} \mathbb{C} \times \{g+z\}$
$v \longmapsto e_g(z) \cdot v$

$\Rightarrow e_g : V \longrightarrow \mathbb{C}^\times$ holo. function $\Rightarrow e_g \in \Gamma(\mathcal{O}_V^*) = H^\times \rightsquigarrow \Lambda$-module

$\Rightarrow (e : \Lambda \longrightarrow H^\times) \in H^1(\Lambda, H^\times)$ as soon as it is a cocycle.

<u>Need:</u> $\delta e = 0$.

$$(\delta e)_{(g,g')} = (g \cdot e_{g'}) \cdot (e_{g \cdot g'})^{-1} \cdot (e_g) \overset{?}{=} 1$$

reverse g & g'.

<u>Note:</u> $\alpha_g \circ \alpha_{g'} = \alpha_{g \cdot g'}$

$$\alpha_{gg'}(t, z) = (e_{gg'}(z) \cdot t, \cancel{z}\ gg' + z)$$

$$\alpha_g(\alpha_{g'}(t, z)) = \alpha_g(e_{g'}(z) \cdot t, z + g')$$

$$= (e_g(z+g')(e_{g'}(z)t), z + gg')$$

<u>So we get:</u>

$$\begin{tikzcd}
L \arrow[d] & H^1(X, \mathcal{O}_X^\times) \arrow[r] \arrow[d] & H^1(\Lambda, H^\times) \arrow[l, bend left, "\cong"] \arrow[d] \\
F & H^2(X, \mathbb{Z}) & H^2(\Lambda, \mathbb{Z}) \arrow[l, "\phi^2"]
\end{tikzcd}$$

ϕ^1

is it the inverse of ϕ^1?

<u>Answer:</u> Yes, it is indeed the inverse of ϕ^1 $\rightsquigarrow$ <u>next time</u>

$$H(z_1, z_2) = F(z_1, z_2) - F(z_2, z_1)$$

$\uparrow$ Hermitian.