Abelian varieties

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Last time:

$/\mathbb{C}$; cpx torus: X cpct conn'd Lie grp $/\mathbb{C}$.

Thm: • $\exp: \underbrace{T_0 X}_{= V} \to X$ Lie grp homom. (univ. cover of X)

• $\underbrace{\ker(\exp)}_{= U} \subseteq V$ discrete subgroup ($u \in U$ $B(u,\varepsilon) \cap U = \{u\}$)

• exp surjective

(↳ in particular, $X \cong \mathbb{C}^n / D$ ← discrete subgroup)

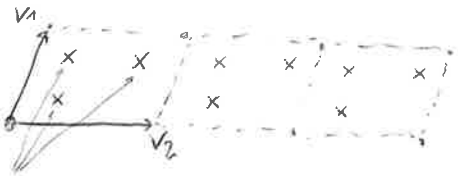
Warm up: what is U?

Lemma: $\Lambda \subseteq \mathbb{R}^n$ discrete subgroup $\Rightarrow$ ftly gen'd.

Pf: WMA $\langle \Lambda \rangle_{\mathbb{R}} = \mathbb{R}^n$ [otherwise replace $\mathbb{R}^n$ by $\langle \Lambda \rangle_{\mathbb{R}^n}$]

$v_1, \dots, v_n \in \Lambda$ $\mathbb{R}$-linear basis

$n = 2$:



finitely many (by discreteness)
moreover, they generate Λ w/ v_1 & v_2.

Cor: $\Lambda \underset{\text{as an abstract group}}{\cong} \mathbb{Z}^r$ (torsion free & ftly gen'd).

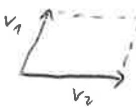
Lemma: $r = \dim_{\mathbb{R}} \langle \Lambda \rangle_{\mathbb{R}}$

Pf: $\underset{}{n = 2}$ "$\dim_{\mathbb{R}} \langle \Lambda \rangle_{\mathbb{R}}$": $\Lambda \underset{\text{as grp}}{\cong} \mathbb{Z}^r$

$v_i \longleftrightarrow e_i$

↯ · Assume $r > 2 \Rightarrow \exists v_3$, $\forall v_3$ not a rtl lattice elt. (as no rational rel. btw e_1, e_2, e_3)

Cover



with balls of radius $\varepsilon > 0$

↳ $\exists$ $n v_3 - (a v_1 + b v_2) \neq m v_3 - (c v_1 + d v_2)$ in the same ball (pigeonhole)

↳ their $\overset{=: y}{\text{difference}}$ has norm $< 2\varepsilon$

↳ $\forall \varepsilon > 0$ $\exists \overset{\neq 0}{y} \in \Lambda$: $|y| < \varepsilon$ ↯ discreteness of Λ. □

$\rightsquigarrow U \cong \mathbb{Z}^{2n}$ ⊛ as abstract groups. A generator set $\{v_1, \dots, v_{2n}\}$ is unique modulo $GL(2n, \mathbb{Z})$. Also, you can assume $v_i = e_i \in \mathbb{C}^n$ for $i = 1, \dots, n$.

$\rightsquigarrow$ <u>Moduli space</u>: $GL(n,\mathbb{C}) \big/ GL(2n, \mathbb{Z}) \ni [v_{n+1}, \dots, v_{2n}]$ ⊗

⊛ <u>Lattice is full rank because</u>: $X \underset{\text{diffeo}}{\cong} \mathbb{R}^{2n} / \Lambda \cong \mathbb{Z}^r$, $r \leq 2n$.

<u>Choose</u>: basis of $\mathbb{R}^n$ s.t. first $v_1, \dots, v_r$ form a basis of Λ, $v_{r+1}, \dots, v_{2n}$ arbitrary. Then

$\Rightarrow X \overset{\text{diffeo}}{\cong} \mathbb{R}^{2n} \Big/ \sum_{i=1}^{r} \mathbb{Z} e_i \cong (S^1)^r \times \mathbb{R}^{2n-r} \overset{X \text{ cpct}}{\Longrightarrow} r = 2n$ $\square$

⊗ <u>Homework</u>: What is the action here of $GL(2n, \mathbb{Z})$ on $GL(n, \mathbb{C})$?

$\boxed{\text{Thm}}$ (Appel – Humbert): Notation: $\mathbb{C}^n \cong V = T_0 X \xrightarrow{\pi} X$ univ. cover ($\mathbb{C}$-linear), $\ker \pi = U \cong \mathbb{Z}^{2n}$ as groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}(U, \mathbb{C}_1^*) & \longrightarrow & \{(H, \alpha) \mid \dots\} & \longrightarrow & \left\{\begin{array}{c}\text{group } H: V \times V \to \mathbb{C} \\ \text{of hermitian forms s.t.} \\ (\operatorname{Im} H)(U \times U) \subseteq \mathbb{Z}\end{array}\right\} & \longrightarrow & 0 \\
 & & \updownarrow & & \updownarrow & & \updownarrow & & \\
0 & \longrightarrow & \operatorname{Pic}^0(X) & \longrightarrow & \operatorname{Pic} X & \xrightarrow{c_1} & \operatorname{Ker}(H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)) & \to & 0
\end{array}$$

$\mathbb{C}_1^* = \{z \in \mathbb{C} \mid |z| = 1\} \cong_{\text{topo}} S^1$

$\operatorname{Hom}(U, \mathbb{C}_1^*)$, $U \cong \mathbb{Z}^{2n} \cong \bigoplus \mathbb{Z} v_i$: $\cong (S^1)^{2n}$, $\alpha \longmapsto (\alpha(v_1), \alpha(v_2), \dots)$

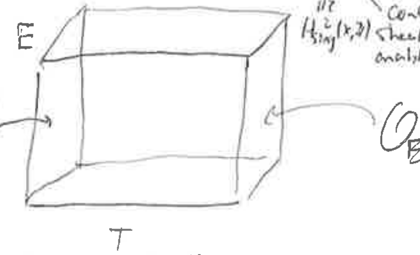
$\operatorname{Pic}^0(X)$: topologically trivial holo. line bdls

$\operatorname{Pic} X$: holo line bdls

$H^2(X, \mathbb{Z}) \cong H^2_{\text{sing}}(X, \mathbb{Z})$, $\mathbb{Z}$: constant sheaf in analytic top.

$\mathcal{O}_X$: holo. structure sheaf

[E ell. curve: $\mathcal{O}_E(p-q) \xrightarrow{\text{deforms holomorphicaly}} \mathcal{O}_E$, $p \neq q$



$\Rightarrow \mathcal{O}_E(p-q)$ is topologically the same as $\mathcal{O}_E$]

<u>Rem</u>: The bijections $\updownarrow$ are explicit

<u>Bottom row</u>: $0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$ exponential sequence, $f \longmapsto e^{2\pi i f}$

(sheaves of abelian groups)

$\Rightarrow H^1(X, \mathbb{Z}_X) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow \overbrace{H^1(X, \mathcal{O}_X^*)}^{\cong \operatorname{Pic} X} \longrightarrow H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$

$\cong \operatorname{Pic} X$ via Čech cohomology: 1 Čech cocycle = transition fcts of holo line bdl

$\Rightarrow 0 \longrightarrow \overbrace{\operatorname{Coker}(\underbrace{H^1(X, \mathbb{Z}_X)}_{\text{we will show: } \cong \mathbb{Z}^{2n}} \to \underbrace{H^1(X, \mathcal{O}_X)}_{\cong \mathbb{C}^n \text{ Book (Hodge theory)}})}^{\text{dual torus} :=} \to \overbrace{\operatorname{Pic} X}^{\cong \text{ holo line bdls}} \xrightarrow{c_1} \overbrace{\ker(H^2(X, \mathbb{Z}) \to H^2(\mathcal{O}_X))}^{\text{topological line bdls}} \to 0$

holo line bdls $\xmapsto{\text{obvious map}}$ topological line bdls

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$\boxed{\text{Key to the pf}}$: $H^i(X, \mathcal{F}_i)$ $\longrightarrow$ $H^i(V, \pi^*\mathcal{F}_i) = 0$ for $i > 0$ for our sheaves ($\mathbb{Z}_X, \mathcal{O}_X, \mathcal{O}_X^*$)

$\Rightarrow H^0(V, \pi^*\mathcal{F}_i) = \Gamma(V, \pi^*\mathcal{F}_i)$

U acts on it w/ isom's of ab grp's $=:$ U-module

$$\Rightarrow H^i(X, \mathcal{F}_i) \overset{\phi^i}{\cong} \underbrace{H^i(U, \overbrace{\Gamma(V, \pi^*\mathcal{F}_i)}^{U\text{-module}})}_{\text{group cohom.}}$$

Next goal: def. of ϕ^i & isoms.

Preparation: group cohom. M is a U-module [$\Leftrightarrow$ M is $\mathbb{Z}[U]$-module]

$$H^i(U, M) \overset{\text{def 1}}{=} \operatorname{Ext}^i_{\mathbb{Z}[U]}(\underset{\substack{\text{trivial}\\ \text{action of } U}}{\mathbb{Z}}, M)$$

$$\overset{\text{def 2}}{=} h^i(\cdots \to C^j \to C^{j+1} \to \cdots), \quad C^j = \operatorname{Hom}_{set}(U^{\oplus j}, M)$$

$\delta^j : C^j \longrightarrow C^{j+1}$

$f \longmapsto \delta f$

$$(\delta f)(g_0, \dots, g_{j+1}) := g_0 \cdot f(g_1, \dots, g_{j+1}) + \sum_{i=0}^{j-1} (-1)^{i+1} f(g_0, \dots, g_i g_{i+1}, \dots, g_{j+1}) + (-1)^{j+2} f(g_0, \dots, g_{j-1})$$

$\boxed{\text{Homework}}$: Show def 1 = def 2.

def 3: derived functor of

$$\underset{\{U\text{-modules}\}}{M} \longmapsto \underset{\{\text{ab grps}\}}{M^U} = \{m \in M \mid \forall u \in U: u \cdot m = m\}$$

$\pi: V \longrightarrow X$, $U = \ker \pi$

$$\underset{\text{Sheaf on } X}{\pi^*\mathcal{F}_i} \longrightarrow \underset{U\text{-module}}{\Gamma(V, \pi^*\mathcal{F}_i)} \rightsquigarrow \Gamma(V, \pi^*\mathcal{F}_i)^U = \Gamma(X, \mathcal{F}_i)$$

$\Rightarrow \Gamma(X, -) \equiv (-)^U \circ \Gamma(V, \pi^*(-))$ as functors

one has to verify that this functor maps injectives to ~~[illegible]~~ acyclics (here injective even)

$\underset{\substack{\text{general}\\ \text{nonsense}}}{\Longrightarrow}$ $R\Gamma(X, -) = R(-)^U \circ R\Gamma(V, \pi^*(-))$ $\underset{\substack{H^i(V, \pi^*\mathcal{F}_i) = 0\\ i>0}}{\Longrightarrow}$ ϕ^j isom.

ϕ^j explicitly: $\phi^j : H^j(U, \Gamma(V, \pi^*\mathcal{F}_i)) \longrightarrow H^j(X, \mathcal{F}_i)$

$f \in C^j = \operatorname{Hom}_{set}(U^{\oplus j}, \Gamma(V, \pi^*\mathcal{F}_i)) \longmapsto (\phi^j f)$

<u>Fix:</u> "nice" cover ~~of~~ $\{V_i\}_{i\in I}$ of X

next time

$$\underbrace{(\phi^j f)_{i_0,\dots,i_j}}_{\in \mathcal{F}(\underbrace{\bigcap_{\nu=0}^{j} V_{i_\nu}}_{=V_{i_0,\dots,i_\nu}})} = f(g_{i_0 i_1}, \dots, g_{i_{j-1} i_j})\big|_{V_i}$$

next time

lying over V_i