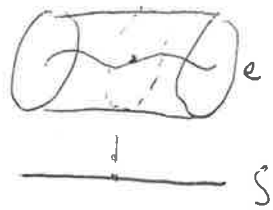
Abelian varieties

$\boxed{\text{Def}}$: $\begin{matrix}\mathcal{X}\\ \downarrow\\ S\end{matrix}$ scheme over S is abelian if it is a projective (? connected ?) group scheme $/S$.

group scheme: group object in the category of $(S\text{-})$ schemes.

What does this mean: • identity elt: $\begin{tikzcd} \mathcal{X} \arrow[d, "f"'] \\ S \arrow[u, bend right, "e"'] \end{tikzcd}$, $f \circ e = \mathrm{Id}_S$ $\rightsquigarrow$



• multiplication: $m : \mathcal{X} \times_S \mathcal{X} \longrightarrow \mathcal{X}$ morphism over S with the expected properties of group multiplication, e.g. associativity:

$$\begin{tikzcd} \mathcal{X} \times_S \mathcal{X} \times_S \mathcal{X} \arrow[r, "\mathrm{Id} \times m"] \arrow[d, "m \times \mathrm{Id}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{X} \times_S \mathcal{X} \arrow[d, "m"] \\ \mathcal{X} \times_S \mathcal{X} \arrow[r, "m"'] & \mathcal{X} \end{tikzcd}$$

. identity property:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "e \times \mathrm{Id}"] \arrow[rr, bend right, "\mathrm{Id}"'] & \mathcal{X} \times_S \mathcal{X} \arrow[r, "m"] \arrow[d, phantom, "\circlearrowright"] & \mathcal{X} \\ & {} & \end{tikzcd}$$

. inverse morph: $i : \mathcal{X} \longrightarrow \mathcal{X}$

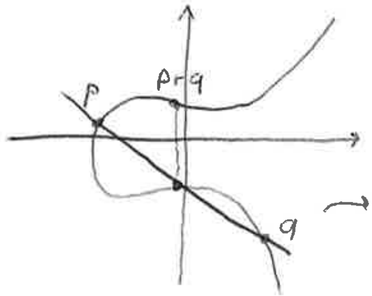
$$\begin{tikzcd} \mathcal{X} \arrow[r, "\mathrm{Id} \times i"] \arrow[dr, "f"'] & \mathcal{X} \times_S \mathcal{X} \arrow[r, "m"] & \mathcal{X} \\ & S \arrow[ur, "e"'] & \end{tikzcd}$$

etc.

Think about $S = \mathbb{C}$:

Examples of group schemes $/\mathbb{C}$: $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $PGL(n, \mathbb{C})$, $(\mathbb{A}^1, +) = \mathbb{G}_a$, $(\mathbb{A}^1 \setminus \{0\}, \cdot) = \mathbb{G}_m$

classical projective example: elliptic curve



$\longrightarrow$ this operation is in fact algebraic.

(i.e. comes from a morphism)

$\boxed{\text{Def}}$: abelian variety = abelian scheme / field or alg. closed field.

<u>Future</u>: ab. variety / $\mathbb{C}$ is smooth.

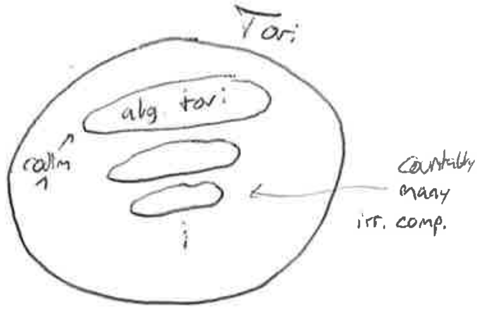
$\boxed{\text{Def}}$: X is a complex torus if it is a compact connected complex Lie group.
(i.e. group object in the category of complex manifolds)

$\boxed{\text{Goal 1}}$: <u>1a</u>: classify tori

<u>1b</u>: criterion for when they are algebraic

$\boxed{\text{Thm}}$ (Chow): $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ closed analytic subset (i.e. defined by hol. fct.'s)
$\Rightarrow$ it is algebraic.

↳ <u>1c</u>: understanding line bdls on tori.



$\boxed{\text{Background}}$ (on 1-param. subgroups, exp map):

G always a complex Lie group (compact, connected).

$\boxed{\text{Def}}$: A <u>1-parameter subgroup</u> is a morphism of Lie groups $(\mathbb{C},+) \longrightarrow G$
(i.e. morphism of groups & of complex mfd's simultaneously).

$\boxed{\text{Thm}}$: $\forall v \in T_e G \ \exists! \gamma_v$ s.t. $\gamma_v'(0) = v$

$\boxed{\text{Ex}}$: $G = \mathbb{G}_m$, $\gamma = e^{c \cdot x} \Rightarrow \gamma'(0) = c$.

$\boxed{\text{Pf}}$ (outline): $\gamma(t_0 + \varepsilon) = L_{\gamma(t_0)} \circ \gamma(\varepsilon)$. ← If there is such a γ

left mult. by $\gamma(t_0)$

↳ $\gamma'(t_0) = T_e L_{\gamma(t_0)}(v)$, $v = \gamma'(0)$.

$G \ni g \longmapsto T_e L_g(v)$ is a left-invariant vector field.

γ = integral curve of this vector field through e. $\square$

$\boxed{\text{Def}}$: $\exp: T_e G \longrightarrow G$ morph. of complex manifolds, defined by $\exp(v) = \gamma_v(1)$.

$\boxed{\text{Note}}$: $\exp(tv) = \gamma_v(t)$

$\boxed{\text{Pf}}$: $\exp(tv) = \gamma_{tv}(1) \rightsquigarrow$ <u>need</u>: $\gamma_v(t) = \gamma_{tv}(1)$

<u>more generally</u>: $\gamma_v(t \cdot s) = \gamma_{tv}(s)$

↳ both $\gamma_v(t \cdot -)$ and $\gamma_{tv}(-)$ are 1-param. subgroups, so by unicity of the above thm, it is enough to show that their derivatives at $s=0$ agree
↳ use chain rule. $\square$

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Linus Rösler

Abelian varieties

$\boxed{\text{Thm}}$: $T_0 \exp$ is invertible $\Rightarrow$ $\exp$ is a local biholomorphism

Goal 1: 1.a.i
- $\exp$ is the universal cover
- $\exp$ is a gp homom.
- ker is a lattice (discrete, in this case full rank subgrp of $(\mathbb{C}^n, +)$)

from now: G compact & connected

$\boxed{\text{Ex}}$: $G = \mathbb{C}^n / \Lambda$, ex: $n=1$: $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}i \sim \{ Zy^2 = (x-1)x(x+1) \} \subseteq \mathbb{P}^2_{\mathbb{C}}$

naturally a complex mfd, as $\mathbb{Z} \oplus \mathbb{Z}i$ discrete

$\boxed{\text{Rem}}$: in general, for $n=1$, $G = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\cdot\tau$ (rotate one vector to 1)

Application: $\mathrm{Aut}(G, e)$ (automorphisms as alg. gps (same by GAGA))
↳ for $\tau = i$ 4 automorphisms, for $\tau = e^{\frac{2\pi i}{6}}$ 6 auto's. (exc)

$\boxed{\text{Exercise}}$. For all other 1 dim'l complex tori: $|\mathrm{Aut}(G, e)| = 2$.

$\boxed{\text{Realizing Goal 1.a.i}}$:

$\boxed{\text{Prop}}$: G (cpct & connected) abelian as a grp.

$\boxed{\text{Pf}}$: $c_x : y \longmapsto xyx^{-1}$ conjugation by x $\rightsquigarrow$ morph. of holo. mfd's.

$(\mathrm{Ad}: x \in G \longmapsto c_x \in \mathrm{Aut}(G)) \rightsquigarrow x \in G \overset{\phi}{\longmapsto} T_e c_x \in \mathrm{Aut}_{\mathbb{C}}(T_e G) \subseteq GL(n, \mathbb{C}) \subseteq \mathbb{C}^{n \times n}$

↳ ϕ_{ij} the coordinate functions: $\underset{\text{cpct}}{G} \xrightarrow[\text{holo}]{} \mathbb{C} \Rightarrow \phi_{ij} = \text{const.} \Rightarrow \phi$ const.

Note: $c_e = \mathrm{Id}_G \Rightarrow T_e c_e = \mathrm{Id}_{T_e G} \Rightarrow \phi \equiv \mathrm{Id}_{T_e G}$

↳ $\forall x \in G$, $T_e c_x = \mathrm{Id}_{T_e G}$

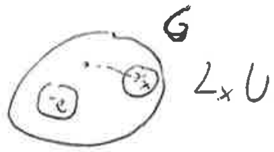
Need: Bootstrapping to $c_x = \mathrm{Id}$. Fact:

$$\begin{tikzcd} T_e G \arrow[r, "T_e c_x"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\circlearrowleft"] & T_e G \arrow[d, "\exp"] \\ G \arrow[r] & G \end{tikzcd}$$

Exc: show this by unicity of 1-param sbgps

$\Rightarrow C_x \circ \exp = \exp \quad \underset{\text{exp is local biholo}}{\Longrightarrow} \quad C_x\big|_{\underbrace{\text{some nbhd of } e}_{=U}} = \mathrm{Id}$

$\boxed{\text{Exc}}$: U generates G. Idea: open and closed



$\Rightarrow C_x = \mathrm{Id}$. $\square$

$\boxed{\text{Cor}}$ exp is a homomorphism.

$\boxed{\text{Pf}}$: $t \longmapsto \exp(t \cdot x) \cdot \exp(ty) =: \gamma(t)$

$$t_1 + t_2 \longmapsto \exp(t_1 x)\exp(t_2 x)\exp(t_1 y)\exp(t_2 y) = \gamma(t_1+t_2)$$

$\| \; G$ comm.

$$\underbrace{\exp(t_1 x)\exp(t_1 y)}_{\gamma(t_1)} \; \underbrace{\exp(t_2 x)\exp(t_2 y)}_{\gamma(t_2)}$$

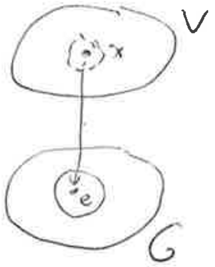
↳ γ 1-param. subgrp $\Rightarrow \gamma(t) = \exp(tz)$, where $z = \gamma'(0) = x+y$

$\Rightarrow$ plug-in $t=1$: $\exp(x+y) = \exp(x)\exp(y)$. $\square$

$$\begin{array}{l} V = \mathbb{C}^n \\ \big\downarrow \exp \\ G \end{array}$$

Lie grp homom.

local biholo. (everywhere, by translating origin).



$\Rightarrow \mathrm{Ker}(\exp) = U$ is a discrete subgroup.

Also, $\mathrm{Im}(\exp) \supseteq$ a nbhd of origin

↳ contains sbgrp generated by the nbhd

↳ exp surjective.