Homological Algebra Seminar Week 9

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0.1 Notions of convergence of spectral sequences

The main goal of this lecture is to extend the notion of convergence of a spectral sequence to the unbounded case. We first state the following example which will be relevant in the second half of these notes.

Example 0.1. Let E be a 1st quadrant spectral sequence, i.e. $E^a_{pq} = 0$ whenever p or q is negative. Suppose that $E^a_{pq} \Rightarrow H_{p+q}$. Then the filtration on H_n is

$$0 = F_{-1}H_n \subset \cdots \subset F_nH_n = H_n.$$

The morphisms (on the E^r page) whose codomain is on the x-axis, or whose domain is on the y-axis must be the 0 map, as a direct consequence of being a first quadrant spectral sequence. In particular this implies $E^r_{p0} \subset E^a_{p0}$ and $E^a_{0q} \hookrightarrow E^r_{0q}$. And so in particular we have for the E^∞-page $E^\infty_{p0} \subset E^a_{p0}$ and $E^a_{0q} \hookrightarrow E^\infty_{0q}$.
By definition of convergence, this implies we have maps $H_n \to E^\infty_{n0} \subset E^a_{n0}$ and $E^q_{0n} \hookrightarrow E^\infty_{0q} \subset H_n$. These morphisms are called *edge homomorphisms*. Because historically spectral sequences first appeared in the case of fiber sequences of topological spaces we call the E^r_{0q} *fiber terms* and we call the E^r_{p0} *base terms*.

The terminology of fiber and base terms will be retroactively motivated in the next section. We now introduce our first notion of convergence.

Definition 0.2. We say that a spectral sequence E weakly converges to H_* if we are given $H_n \in \mathrm{Ob}(\mathcal{A})$ each having a filtration

$$\cdots \subset F_{p-1}H_n \subset F_pH_n \subset \cdots \subset H_n$$

such that for all integer p, q there are isomorphisms $\beta_{pq} : E^\infty_{pq} \to F_pH_{p+q}/F_{p-1}H_{p+q}$.

In order to introduce our second, better, notion of convergence, we introduce the following terminology on filtrations.

Definition 0.3. Let $H \in \mathrm{Ob}(\mathcal{A})$ with a filtration

$$\cdots \subset F_{p-1}H \subset F_pH \subset \cdots \subset H.$$

We call this filtration *exhaustive* if $\bigcup_p F_pH = H$, *Hausdorff* if $\bigcap_p F_pH = 0$ and *complete* if $H \cong \varprojlim H/F_pH$.

With this in hand, we are able to state the following notion of convergence.

Definition 0.4. We say E *approaches* (or abuts to) H_* if it weakly converges to H_* with filtrations

$$\cdots \subset F_{p-1}H_n \subset F_pH_n \subset \cdots \subset H_n$$

that are exhaustive and Hausdorff.

Remark 0.5. If E weakly converges to H_*, one can observe that it must abut to $\bigcup_p F_pH_*/\bigcap_p F_pH_*$.

The final notion of convergence is the following.

Definition 0.6. We say that E *converges* to H_* if E abuts to H_*, E is regular and the filtrations of H_* are complete.

We introduced the three notions of convergence from weakest to strongest, luckily for nice enough spectral sequences we have results such as the following, which allows us to promote weaker notions of convergence.

Lemma 0.7. *Let E be a spectral sequence that is bounded below. Then E approaches H_* if and only if E converges to H_*.*

Proof. ($\Leftarrow$) This is immediate as the definition of convergence includes "approaches to H_*".
($\Rightarrow$) Bounded below implies regularity by a lemma from the previous week. For all n there exists s such that $E^a_{pq} = 0$ for all $p < s$ and such that $p + q = n$ by definition of bounded below. This implies that the same holds at the E^∞-page which by definition of approaching implies that $F_pH_n/F_{p-1}H_n = 0$. This in turn implies that for all $p < s$ we have that $F_pH_n = \bigcap_p F_pH_n$, which is equal to 0 because the filtration is Hausdorff by assumption. This implies that $H_n \cong \varprojlim H_n/F_pH_n$, i.e. that the filtration is regular. This concludes the proof. $\square$

Even when a spectral sequence converges, it need not give complete information. Thus, it can be useful to have other methods to relate the information on the pages of the spectral sequence with the abutment. In line with categorical thinking, a first attempt at this is the following definition.

Definition 0.8. Let E, E' be two spectral sequences that converge to H_*, H'_* respectively with isomorphisms $\beta_{pq} : E^\infty_{pq} \to F_pH_{p+q}/F_{p-1}H_{p+q}$ and isomorphisms $\beta'_{pq} : E'^\infty_{pq} \to F_pH'_{p+q}/F_{p-1}H'_{p+q}$.
We call a morphism $h : H_* \to H'_*$ compatible with a morphism $f : E \to E'$ if h preserves the filtration (i.e. $h(F_pH_n) \subset F_pH'_n$) such that the following diagram commutes

$$\begin{array}{ccc} E^\infty_{pq} & \xrightarrow{\beta_{pq}} & F_pH_{p+q}/F_{p-1}H_{p+q} \\ {\scriptstyle f^\infty_{pq}}\downarrow & & \downarrow{\scriptstyle \bar{h}} \\ E'^\infty_{pq} & \xrightarrow{\beta'_{pq}} & F_pH'_{p+q}/F_{p-1}H'_{p+q} \end{array} .$$

The above definition is used in the following theorem to assist in spectral sequence computations.

Theorem 0.9. *(Comparison theorem) Let E, E' converge to H_*, H'_* respectively, and $h : H_* \to H'_*$ be a morphism which is compatible with $f : E \to E'$. If for all p, q there is an r such that $f^r_{pq} : E^r_{pq} \to E'^r_{pq}$ is an isomorphism, then the map $h : H_* \to H'_*$ is an isomorphism.*

Proof. By the mapping lemma f^∞_{pq} is an isomorphism. Weak convergence and compatibility of h and f yields (via a simple application of the third isomorphism theorem) that for all s, p we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_{p-1}H_n/F_sH_n \arrow[r] \arrow[d] & F_pH_n/F_sH_n \arrow[r] \arrow[d] & E^\infty_{p,n-p} \arrow[r] \arrow[d, "f^\infty_{p,n-p}"] & 0 \\
0 \arrow[r] & F_{p-1}H'_n/F_sH'_n \arrow[r] & F_pH'_n/F_sH'_n \arrow[r] & E'^\infty_{p,n-p} \arrow[r] & 0 .
\end{tikzcd}$$

Fixing s, we get by induction on p and by the 5 lemma that $F_pH_n/F_sH_n \cong F_pH'_n/F_sH'_n$ for all p. By exhaustivity, this implies that we have an isomoprhism $H_n/F_sH_n \cong H'_n/F_sH'_n$, we conclude taking inverse limits (with respect to s) by completeness of the filtration. ☐

Remark 0.10. A spectral sequence E may converge to two different limits, so H_* can be difficult to reconstruct from this data.

Example 0.11. Let E be a first quadrant spectral sequence such that $E^\infty_{pq} \cong \mathbb{Z}/2\mathbb{Z}, \forall p, q \geq 0$. Then without further information, we do not know whether H_2 is $\mathbb{Z}/8\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^{\otimes 3}$. The comparison theorem can help us in situations such as these.

0.2 The Lerray-Serre spectral sequence

The goal of this section is to define the Lerray-Serre spectral sequence and show how it can be useful for computing homology groups of spaces.

Definition 0.12. Let Z be a pointed topological space and $I = [0, 1]$ the unit interval. We say that a morphism $f : X \to Y$ of pointed topological spaces satisfies the *homotopy lifting property* for Z if given a commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "-\times 0"] & X \arrow[d, "f"] \\
Z \times I \arrow[r, "H"'] & Y
\end{tikzcd}$$

there always exists $G : Z \times I \to X$ such that

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "-\times 0"] & X \arrow[d, "f"] \\
Z \times I \arrow[r, "H"'] \arrow[ur, "G"'] & Y
\end{tikzcd}$$

commutes.

Definition 0.13. We call a sequence of composable maps $F \xrightarrow{\iota} E \xrightarrow{\pi} B$ in the category of pointed topological spaces a Serre fibration if ι is the inclusion of $\pi^{-1}(*_B)$, the fiber of π over the base point of B, into E and π satisfies the homotopy lifting property for all CW-complexes.

Example 0.14. For B a pointed topological spaces we have a Serre fibration $\Omega B \xrightarrow{\iota} B^I \xrightarrow{ev_1} B$, where B^I is the space of maps $I \to B$ starting at the base point of B, with the compact open topology and $ev_1 : B^I \to B$ is the map given by evaluating at 1. We omit the proof that $B^I \to B$ satisfies the appropriate homotopy lifting property.

With the appropriate topological and homological language, we can now state the following theorem of Serre.

Theorem 0.15. *(Serre Spectral Sequence) Let $F \xrightarrow{\iota} E \xrightarrow{\pi} B$ be a Serre fibration. Assume further that B is simply connected. Then there is a first quadrant spectral sequence*
$$E^2_{p,q} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(E).$$

Remark 0.16. The edge map $H_q(F) \hookrightarrow E^\infty_{0,q} \subset H_q(E)$ is the map $H_q(\iota)$ and if we also assume that F is connected, the other edge map $H_p(E) \hookrightarrow E^\infty_{p,0}(B)$ is $H_*(\pi)$.

There are many application in homology computations of the above result, we present here a specific one

Proposition 0.17. *With the assumptions of the theorem and further assuming that F is connected, if there exist $n_1 \geq 1$ and $n_2 \geq 2$ such that, for any abelian group A*
$$H_i(F, A) = 0, \forall 0 < i < n_1,$$
$$H_j(B, A) = 0, \forall 0 < j < n_2,$$
then there is a long exact sequence

$$H_{n_1+n_2-1}(F) \xrightarrow{\iota_*} H_{n_1+n_2-1}(E) \xrightarrow{\pi_*} H_{n_1+n_2-1}(B) \longrightarrow H_{n_1+n_2-2}(F) \longrightarrow \cdots .$$

Proof. We look at the Leray-Serre Spectral sequence in this case. On the y-axis we have $E^2_{0q} = H_0(B, H_q(F)) \cong H_q(F)$ and similarly on the x-axis we have $E^2_{p0} = H_p(B, H_0(F)) \cong H_p(B)$ where the isomorphisms follow from the fact that $H_0(F) \cong \mathbb{Z}$ and $H_0(B) \cong \mathbb{Z}$ which are the homological incarnation of the connectivity assumptions on B and F. The vanishing assumptions on the homology of F and B (with arbitrary coefficients) imply, by the universal coefficient theorem, that $E^2_{pq} = 0$ for $p < n_2$ or $q < n_1$ and one of them non-zero.

We get vanishing of the corresponding terms of the E^∞ page. In particular, this implies that the filtration quotients of F_pH_k are trivial for $1 \leq k \leq n_1 + n_2 - 1$, as under these conditions, when $k = p + q$, $p < n_2$ or $q < n_1$ and at least one of p and q is non zero, thus $F_pH_{p+q}/F_{p-1}H_{p+q} \cong E^\infty_{pq} \cong 0$. This in particular implies that in this range we have isomorphism $F_0H_k(E) \cong E^\infty_{0,k}$ and $E^\infty_{k,0} \cong H_k(E)/E^\infty_{0,k}$, which follow from definition of convergence. This gives the following short exact sequence

$$0 \to E^\infty_{0,k} \to H_k(E) \to E^\infty_{k,0} \to 0.$$

Looking at the kth page, we have a differential $E^k_{k,0} \to E^k_{0,k-1}$, by turning the page and recalling that we are working with a first quadrant spectral sequence we see that this map fits in the following exact sequence

$$0 \to E^{k+1}_{k,0} \to E^k_{k,0} \to E^k_{0,k-1} \to E^{k+1}_{0,k-1} \to 0.$$

For first quadrant spectral sequences, one can notice that $E^{k+1}_{k,0} \cong E^\infty_{k,0}$ and $E^{k+1}_{0,k-1} \cong E^\infty_{0,k-1}$ because all the differentials that can modify these groups live on earlier pages. Similarly, one can notice that $H_k(B) \cong E^2_{k,0} \cong E^k_{k,0}$ and $H_{k-1}(F) \cong E^2_{0,k-1} \cong E^k_{0,k-1}$ by using that $E^2_{pq} = 0$ for $p < n_2$ or $q < n_1$ and one of them non-zero. This allows to rewrite the above exact sequence as

$$0 \to E^\infty_{k,0} \to H_k(B) \to H_{k-1}(F) \to E^\infty_{0,k-1} \to 0.$$

We can splice this exact sequence with the following short exact sequence

$$0 \to E^\infty_{0,k} \to H_k(E) \to E^\infty_{k,0} \to 0$$

in order to obtain the desired long exact sequence

$$H_{n_1+n_2-1}(F) \xrightarrow{\iota_*} H_{n_1+n_2-1}(E) \xrightarrow{\pi_*} H_{n_1+n_2-1}(B) \longrightarrow H_{n_1+n_2-2}(F) \longrightarrow \cdots .$$

$\square$

Remark 0.18. We can of course extend the above long exact sequence "naively" by $0 \to \ker(H_{n_1+n_2-1}(F) \xrightarrow{\iota_*} H_{n_1+n_2-1}(E))$. It would be nice to have a more direct understanding of how to extend the above long exact sequence. Inspecting the spectral sequence once more, we see that a potential obstruction comes from the image of $E^{n_1}_{n_2,n_1}$ in $H_{n_1+n_2-1}(F)$. By the universal coefficient theorem, we have in general

$$E^2_{p,q} = H_p(B, H_q(F)) \cong H_p(B) \otimes H_q(F) \oplus \mathrm{Tor}^{\mathbb{Z}}_1(H_{p-1}(B), H_q(F)).$$

In particular, due to vanishing assumptions, in the context of interest to us the first obstruction to a long exact sequence in homology should be related to $H_{n_2}(B, H_{n_1}(F)) \cong H_{n_2}(B) \otimes H_{n_1}(F)$.

Remark 0.19. We resume the example of the path space fibration $\Omega B \to B^I \to B$. Recall that B is simply connected, so that, ignoring differentials and only making explicit the terms we are interested in, the E_2 page will look like

$$
\begin{array}{|ccccc}
H_2(\Omega B) & E^2_{12} & E^2_{22} & E^2_{32} & E^2_{42} \\
H_1(\Omega B) & H_1(B, H_1(\Omega B)) & H_2(B, H_1(\Omega B)) & E^2_{31} & E^2_{41} \\
\mathbb{Z} & E^2_{10} & H_2(B) & H_3(B) & H_4(B) \\
\hline
\end{array} .
$$

We know that this spectral sequence converges to the homology of a contractible space. This in particular means that $H_2(B, H_1(\Omega B))$ has to vanish after turning to the E_3 page, in particular we have an exact sequence

$$H_4(B) \to H_2(B, H_1(\Omega B)) \to H_2(\Omega B).$$

The cokernel of the last map is E^3_{02} and we see, due to lack of space for further differentials, that it must be killed upon turning to the 4th page, so that it must be isomorphic to E^3_{30}. However, by the universal coefficient theorem and because B is simply connected we have $H_1(B, H_1(\Omega B)) \cong 0$, so that $E^3_{30} \cong E^2_{30} \cong H_3(B)$. This gives us an exact sequence

$$H_4(B) \to H_2(B, H_1(\Omega B)) \to H_2(\Omega B) \to H_3(B) \to 0.$$

Because the term $H_2(B, H_1(\Omega B))$ is a bit awkward we would like to make it a bit clearer. We first do this via an application of the universal coefficient theorem. Before writing out what this yields, we notice that a combination of the Hurewicz isomorphism and the isomorphism $\pi_i(\Omega X) \cong \pi_{i+1}(X)$ gives us

$$H_1(\Omega X) \cong \pi_1(\Omega X)^{ab} \cong \pi_2(B) \cong H_2(B).$$

Now an application of the universal coefficient theorem gives

$$H_2(B, H_1(\Omega B)) \cong H_2(B, H_2(B))$$

$$\cong H_2(B) \otimes H_2(B) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_1(B), H_2(B)) \cong H_2(B) \otimes H_2(B).$$

Putting all of this together gives us an exact sequence

$$H_4(B) \to H_2(B) \otimes H_2(B) \to H_2(\Omega B) \to H_3(B) \to 0.$$