Homological Algebra Seminar Week 4

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1 Long Exact Sequences

Theorem 1.1. *Let $\mathcal{A}$ be an abelian category, and suppose that*

$$0 \longrightarrow A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \longrightarrow 0$$

is a short exact sequence in $\mathrm{Ch}(\mathcal{A})$. *Then there is a collection of* natural[1] *maps* $\{\partial_n : H_n(C) \to H_{n-1}(A)\}_{n\in\mathbb{Z}}$, *which we call* connecting homomorphisms, *such that*

$$\cdots \xrightarrow{\tilde{g}_{n+1}} H_{n+1}(C) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{\tilde{f}_n} H_n(B) \xrightarrow{\tilde{g}_n} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{\tilde{f}_{n-1}} \cdots$$

is an exact sequence in $\mathcal{A}$, where for each $n \in \mathbb{Z}$, $\tilde{f}_n$ (respectively, $\tilde{g}_n$) is the image of the chain map $f_\bullet$ (respectively, $g_\bullet$) under the functor $H_n : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$.

Before proving this theorem, we state the *Snake Lemma*, which will help in our construction of the connecting homomorphisms $\partial_n : H_n(C) \to H_{n-1}(A)$.

Lemma 1.2 (The Snake Lemma)**.** *Let $\mathcal{A} = R$–**mod** for some ring R, and suppose that we have a commutative diagram in $\mathcal{A}$ of the form*

$$\begin{tikzcd}
 & A' \arrow[r, "p_1"] \arrow[d, "f"] & B' \arrow[r, "p_2"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i_1"] & B \arrow[r, "i_2"] & C &
\end{tikzcd} .$$

Then, if the rows of this diagram are exact, there is an exact sequence

$$\ker(f) \xrightarrow{p_1} \ker(g) \xrightarrow{p_2} \ker(h) \xrightarrow{\partial} \mathrm{coker}(f) \xrightarrow{\tilde{\imath}_1} \mathrm{coker}(g) \xrightarrow{\tilde{\imath}_2} \mathrm{coker}(h) \ ,$$

where for $a \in A$, $\tilde{\imath}_1$ maps $a + f(A')$ to $i_1(a) + g(B')$, and for $b \in B$, $\tilde{\imath}_2$ maps $b+g(B')$ to $i_2(b)+h(C')$. Also, we can compute $\partial(c')$ for any $c' \in \ker(h)$. First, we find $b' \in B'$ such that $p_2(b') = c'$, then we find the unique $a \in A$ such that $i(a) = g(b')$. Then ∂ satisfies

$$\partial(c') = a + f(A'). \tag{1}$$

[1] The sense in which the connecting homomorphisms are *natural* is explained in Remark 1.6.

Proof. Please see Exercise 1 of this week's exercise sheet. □

Proof of Theorem 1.1. By the Freyd-Mitchell embedding Theorem, it suffices to consider the case when $\mathcal{A} = R$–**mod** for some ring R. Fix an integer n, and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "f"] \arrow[d, "d^A"] & B_n \arrow[r, "g"] \arrow[d, "d^B"] & C_n \arrow[r] \arrow[d, "d^C"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f"] & B_{n-1} \arrow[r, "g"] & C_{n-1} \arrow[r] & 0
\end{tikzcd},$$

the rows of which are exact by Week 3, Exercise 1. Note that each f_n is injective and each g_n is surjective. Hence we obtain via the Snake Lemma an exact sequence

$$Z_n(A) \xrightarrow{f} Z_n(B) \xrightarrow{g} Z_n(C) \longrightarrow A_{n-1}/dA_n \xrightarrow{\tilde{f}} B_{n-1}/dB_n \xrightarrow{\tilde{g}} C_{n-1}/dC_n \ ,$$

where the morphism $\tilde{f} : A_{n-1}/dA_n \to B_{n-1}/dB_n$ satisfies

$$\tilde{f}(a + dA_n) = f_{n-1}(a) + dB_n$$

for all $a \in A_{n-1}$. The morphism $\tilde{g} : B_{n-1}/dB_n \to C_{n-1}/dC_n$ satisfies an analogous definition and is surjective because g_{n-1} is surjective. Since $n \in \mathbb{Z}$ was arbitrary, we may construct a new commutative diagram

$$\begin{tikzcd}
 & A_n/dA_{n+1} \arrow[r, "\tilde{f}"] \arrow[d, "d_*^A"] & B_n/dB_{n+1} \arrow[r, "\tilde{g}"] \arrow[d, "d_*^B"] & C_n/dC_{n+1} \arrow[r] \arrow[d, "d_*^C"] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r, "f"] & Z_{n-1}(B) \arrow[r, "g"] & Z_{n-1}(C) &
\end{tikzcd} \qquad (2)$$

with exact rows. The morphism $d_*^A : A_n/dA_{n+1} \to Z_{n-1}(A)$ satisfies

$$d_*^A(a + dA_{n+1}) = d_n^A(a)$$

for all $a \in A_n$, and the morphisms d_*^B, d_*^C satisfy analogous relations. To see why the left square in (2) commutes, let $a \in A_n$ and observe that

$$\begin{aligned}
d_*^B \circ \tilde{f}(a + dA_{n+1}) &= d_*^B(f_n(a) + dB_{n+1}) \\
&= d_n^B(f_n(a)) \\
&\overset{!}{=} f_{n-1}(d_n^A(a)) \\
&= f_{n-1} \circ d_*^A(a + dA_{n+1}),
\end{aligned}$$

as needed, where the marked equality holds because $f_\bullet$ is a chain map from $A_\bullet$ to $B_\bullet$. An identical argument shows that the right square commutes. Note that the kernel of the morphism $d_*^A : A_n/dA_{n+1} \to Z_{n-1}(A)$ is

$$\begin{aligned}
\ker(d_*^A) &= \{a + dA_{n+1} : a \in A_n,\ d_n^A(a) = 0\} \\
&= Z_n(A)/dA_{n+1} \\
&= H_n(A),
\end{aligned}$$

and its cokernel is

$$\begin{aligned}\operatorname{coker}(d_*^A) &= Z_{n-1}(A)/\operatorname{im}(d_*^A)\\ &= Z_{n-1}(A)/\{d_n^A(a) : a \in A_n\}\\ &= Z_{n-1}/dA_n\\ &= H_{n-1}(A).\end{aligned}$$

Hence, a second application of the Snake Lemma to (2) yields a connecting homomorphism $\partial_n : H_n(C) \to H_{n-1}(A)$ such that the sequence

$$H_n(A) \xrightarrow{\tilde{f}_n} H_n(B) \xrightarrow{\tilde{g}_n} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{\tilde{f}_{n-1}} H_{n-1}(B) \xrightarrow{\tilde{g}_{n-1}} H_{n-1}(C).$$

is exact. We note that $\tilde{f}_k = H_k(f_\bullet)$ and $\tilde{g}_k = H_k(g_\bullet)$ for $k \in \{n, n-1\}$. Since this sequence is exact for all $n \in \mathbb{Z}$, we may paste them together to obtain the desired long exact sequence. □

Remark 1.3. *Before continuing, we note that in the above proof we can actually calculate the image of* $[c] \in H_n(C)$ *under* ∂_n *using Equation* (1). *Let* $c \in Z_n(C)$. *Let* $b \in B_n$ *be such that* $\tilde{g}_n$ *maps* $[b] \in B_n/dB_{n+1}$ *to* $[c]$, *i.e.,* $[g_n(b)] = [c]$ *in* C_n/dC_{n+1}. *Then there is* $a \in Z_{n-1}(A)$ *such that*

$$f_{n-1}(a) = d_*^B([b]) = d_n^B(b).$$

It follows that the connecting homomorphism $\partial_n : H_n(C) \to H_{n-1}(A)$ *satisfies* $\partial_n([c]) = [a]$.

In our statement of Theorem 1.1, we mentioned that the connecting homomorphisms $\{\partial_n : H_n(C) \to H_{n-1}(A)\}_{n\in\mathbb{Z}}$ are *natural*. We now give the precise meaning of this statement. To do so, we introduce two new categories.

Remark 1.4. *Let* $\mathcal{C}$ *be an abelian category. Then there is category* $\mathcal{S}(\mathrm{Ch}(\mathcal{C}))$ *whose objects are the short exact sequences of chain complexes in* $\mathcal{S}$. *A morphism in* $\mathcal{S}(\mathrm{Ch}(\mathcal{C}))$ *from* $0 \to A \to B \to C \to 0$ *to* $0 \to A' \to B' \to C' \to 0$ *is a commutative diagram of the shape*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0. \end{tikzcd}$$

The category $\mathcal{L}(\mathcal{C})$ *of long exact sequences in* $\mathcal{C}$ *is defined analogously.*

Proposition 1.5. *Let* $\mathcal{C}$ *be an abelian category. Then we may define a functor from* $\mathcal{S}(\mathrm{Ch}(\mathcal{C}))$ *to* $\mathcal{L}(\mathcal{C})$ *by mapping a short exact sequence of chain complexes in* $\mathcal{C}$

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

to the long exact sequence in $\mathcal{C}$

$$\cdots \xrightarrow{\tilde{g}_{n+1}} H_{n+1}(C) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{\tilde{f}_n} H_n(B) \xrightarrow{\tilde{g}_n} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{\tilde{f}_{n-1}} \cdots ,$$

and by mapping a morphism in $\mathcal{S}(\mathrm{Ch}(C))$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "u"] & B' \arrow[r, "v"] & C' \arrow[r] & 0
\end{tikzcd} \tag{3}$$

to the following morphism in $\mathcal{L}(\mathcal{C})$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_n(A) \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\alpha}"] & H_n(B) \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{\beta}"] & H_n(C) \arrow[r, "\partial"] \arrow[d, "\tilde{\gamma}"] & H_{n-1}(A) \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\alpha}"] & \cdots \\
\cdots \arrow[r, "\partial'"] & H_n(A') \arrow[r, "\tilde{u}"] & H_n(B') \arrow[r, "\tilde{v}"] & H_n(C') \arrow[r, "\partial'"] & H_{n-1}(A') \arrow[r, "\tilde{g}"] & \cdots .
\end{tikzcd} \tag{4}$$

Proof. The only non-trivial part of this statement is that the diagram in (4) defines a morphism in $\mathcal{L}(C)$. This holds if each square in the ladder diagram commutes. $\tilde{\beta}_n\tilde{f}_n = \tilde{u}_n\tilde{\alpha}_n$ and $\tilde{\gamma}_n\tilde{g}_n = \tilde{v}_n\tilde{\beta}_n$ follow immediately from the commutativity of the diagram in (3) and the functoriality of H_n. Hence it suffices to show that

$$\begin{tikzcd}
H_n(C) \arrow[r, "\partial_n"] \arrow[d, "\tilde{\gamma}_n"] & H_{n-1}(A) \arrow[d, "\tilde{\alpha}_{n-1}"] \\
H_n(C') \arrow[r, "\partial'_n"] & H_{n-1}(A')
\end{tikzcd} \tag{5}$$

commutes. Let $[c]$ be an arbitrary element of $H_n(C)$, where $c \in Z_n(C)$. By Remark (1.3), if $b \in B_n$ is such that $\tilde{g}_n[b] = [c]$, and $a \in Z_{n-1}(A)$ is such that $f_{n-1}(a) = d_n^B(b)$, then $\partial_n[c] = [a]$. We want to find the image of $\tilde{\gamma}_n[c] \in H_n(C')$ under ∂'_n. Note that $\beta_n(b) \in B'_n$ is such that

$$\tilde{v}_n[\beta_n(b)] = \tilde{v}_n\tilde{\beta}_n[b] = \tilde{\gamma}_n\tilde{g}_n[b] = \tilde{\gamma}_n[c],$$

and $\alpha_{n-1}(a) \in Z_{n-1}(A')$ is such that

$$u_{n-1}(\alpha_{n-1}(a)) = \beta_{n-1}(f_{n-1}(a)) = \beta_{n-1}(d_n^B(b)) = d_n^{B'}(\beta_n(b)).$$

Thus ∂'_n maps $\tilde{\gamma}_n[c]$ to $[\alpha_{n-1}(a)] = \tilde{\alpha}_{n-1}[a] = \tilde{\alpha}_{n-1}\partial_n[c]$, i.e.,

$$\partial'_n\tilde{\gamma}_n[c] = \tilde{\alpha}_{n-1}\partial_n[c],$$

as needed. Hence we have mapped each morphism in $\mathcal{S}(\mathrm{Ch}(\mathcal{C}))$ to a well-defined morphism in $\mathcal{L}(\mathcal{C})$. It is immediately seen that this mapping preserves identity morphisms and compositions. We thus have a functor from $\mathcal{S}(\mathrm{Ch}(\mathcal{C}))$ to $\mathcal{L}(\mathcal{C})$. $\square$

Remark 1.6. *For each integer k, define functors L_k, R_k from $\mathcal{S}(\mathrm{Ch}(\mathcal{C}))$ to $\mathcal{C}$ such that*

$$L_k\Big(0 \to A \to B \to C \to 0\Big) = H_k(A),$$

and

$$R_k\Big(0 \to A \to B \to C \to 0\Big) = H_k(C).$$

Then the commutativity of the diagram in (5) *shows that for each integer n, we may define a natural transformation $\partial_n : R_n \implies L_{n-1}$ by assigning to each short exact sequence $0 \to A \to B \to C \to 0$ of chain complexes in $\mathcal{C}$ the connecting homomorphism $H_n(C) \to H_{n-1}(A)$. It is for this reason that we say the connecting homomorphisms in Theorem 1.1 are natural.*

We now give a two brief examples to demonstrate the usefulness of long exact sequences in algebraic topology.

Example 1.7. If X is a topological space, the singular complex $S(X)$ is the chain complex $C_\bullet(X)$, where for $n \geq 0$, $C_n(X)$ is the free abelian group with all singular n-simplices on X as its basis. Elements of $C_n(X)$ are called n-chains on X, and $H_n(X)$ is defined to be the nth homology group of $S(X)$. If A is a subspace of X, then there is a short exact sequence

$$0 \to S(A) \to S(X) \to S(X, A) \to 0, \tag{6}$$

where $S(X, A)$ is the quotient chain complex $S(X)/S(A)$. Let $H_n(X, A)$ be the nth homology group of $S(X, A)$. Each element of $H_n(X, A)$ is represented a relative n-cycle: an n-chain on X whose boundary, i.e., its image under the differential $C_n(X) \to C_{n-1}(X)$, actually lies in $C_{n-1}(A)$. Via Theorem 1.1, we obtain from (6) a long exact sequence

$$\cdots \to H_n(X) \to H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(X) \to \cdots . \tag{7}$$

It turns out that the connecting homomorphism ∂ takes a relative cycle representing a class of $H_n(X, A)$ to the class in $H_{n-1}(A)$ represented by its boundary.

Example 1.8. Suppose that A, B are subspaces of a topological space X such that $X = \mathrm{int}(A) \cup \mathrm{int}(B)$. Then there is a short exact sequence of chain complexes

$$0 \to S(A \cap B) \to S(A) \oplus S(B) \to S(X) \to 0,$$

which is sent by the functor in Proposition 1.5 to the long exact sequence

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots .$$

This long exact sequence is known as the *Mayer-Vietoris sequence*.

2 Chain Homotopies

Definition 2.1. A chain complex $C_\bullet$ is called *split* if there are morphisms $\{s_n : C_n \to C_{n+1}\}$, called *splitting maps*, such that $d_{n+1} = d_{n+1}s_nd_{n+1}$ for every integer n. A chain complex $C_\bullet$ is *split exact* if it is split and acyclic.

Example 2.2. We show that every chain complex $C_\bullet$ of vector spaces over a field k is split. For each integer n, pick $B'_n \leq C_n$ such that $C_n = Z_n \oplus B'_n$. Note that

$$B'_n \cong C_n/Z_n \cong \operatorname{im}(d_n) = B_{n-1}. \tag{8}$$

Let $\pi_n : Z_n \oplus B'_n \to Z_n$ and $\pi'_n : Z_n \oplus B'_n \to B'_n$ be projections. Then the isomorphism $C_n/Z_n \to B'_n$ is the the map taking $[c] \in C_n/Z_n$ to $\pi'_n(c)$. Also, the isomorphism $C_n/Z_n \to B_{n-1}$ maps $[c] \in C_n/Z_n$ to $d_n(c)$. Similarly, since $B_n = \operatorname{im}(d_{n+1})$ is a subspace of Z_n, we may pick $H'_n \leq Z_n$ such that $Z_n = B_n \oplus H'_n$. Note that

$$H'_n \cong Z_n/B_n = H_n. \tag{9}$$

Let $\rho_n : B_n \oplus H'_n \to B_n$ and $\rho'_n : B_n \oplus H'_n \to H'_n$ be the projections corresponding to this decomposition. Now we define the splitting map $s_n : C_n \to C_{n+1}$ to be the composition

$$C_n \xrightarrow{\pi_n} Z_n \xrightarrow{\rho_n} B_n \xrightarrow{q_n} B'_{n+1} \subseteq C_{n+1},$$

where q_n is from the same family of isomorphisms as in (8). What is the composition $d_{n+1}s_nd_{n+1}$? For an element $x \in C_{n+1}$, we have $d_{n+1}(x) \in B_n \subseteq Z_n$, thus

$$\begin{aligned}
d_{n+1}s_n(d_{n+1}(x)) &= d_{n+1}q_n(d_{n+1}(x)) \\
&= d_{n+1}(\pi'_{n+1}(x)) \\
&= d_{n+1}(\pi_{n+1}(x)) + d_{n+1}(\pi'_{n+1}(x)) \\
&= d_{n+1}(x),
\end{aligned}$$

thus $d_{n+1}s_nd_{n+1} = d_{n+1}$. Hence C is a split chain complex. Next, we determine the conditions under which C is not only split, but split exact. Note that for $x \in C_n$, we have $\rho_n\pi_n(x) = d_{n+1}(y)$ for some $y \in C_{n+1}$, thus

$$\begin{aligned}
d_{n+1}s_n(x) &= d_{n+1}q_n(d_{n+1}(y)) \\
&= d_{n+1}(y) \\
&= \rho_n\pi_n(x)
\end{aligned}$$

thus $d_{n+1}s_n$ is projection $C_n \to B_n$. Similarly, one may show that $s_{n-1}d_n$ is projection $C_n \to B'_n$. Since we have the decomposition $C_n = B_n \oplus H'_n \oplus H_n$, it follows from (9) that both the kernel and cokernel of the chain map $ds + sd : C_\bullet \to C_\bullet$ are the trivial homology complex $H_\bullet(C)$, that is, the complex with zero differentials whose nth object is $H_n(C)$. One may argue from here that $C_\bullet$ is exact if and only if $ds + sd$ is the identity chain map.

Remark 2.3. *Given chain complexes* $C_\bullet$ *and* $D_\bullet$*, and* any *collection of morphisms* $\{s_n : C_n \to D_{n+1}\}_{n\in\mathbb{Z}}$*, let* $f_n : C_n \to D_n$ *be the morphism*

$$f_n = d^D_{n+1} s_n + s_{n-1} d^C_n.$$

Then $f_\bullet : C_\bullet \to D_\bullet$ *is in fact a chain map, since for any* $n \in \mathbb{Z}$*:*

$$\begin{aligned} d^D_n f_n &= d^D_n d^D_{n+1} s_n + d^D_n s_{n-1} d^C_n \\ &= d^D_n s_{n-1} d^C_n \\ &= s_{n-2} d^C_{n-1} d^C_n + d^D_n s_{n-1} d^C_n \\ &= (s_{n-2} d^C_{n-1} + d^D_n s_{n-1}) d^C_n \\ &= f_{n-1} d^C_n. \end{aligned}$$

We give a special name to chain maps that are of this form.

Definition 2.4. A chain map $f_\bullet : C_\bullet \to D_\bullet$ is *null homotopic* if there are morphisms $s_n : C_n \to D_{n+1}$, $n \in \mathbb{Z}$, such that

$$f_n = d^D_{n+1} s_n + s_{n-1} d^C_n$$

for every integer n. The maps $\{s_n\}_{n\in\mathbb{Z}}$ are called a *chain contraction* of f. If the identity morphism $C_\bullet \to C_\bullet$ of a chain complex $C_\bullet$ is null homotopic, we say that $C_\bullet$ is *contractible*.

Returning to Example 2.2, we see that a chain complex $C_\bullet$ of vector spaces over a field k is split exact if and only if $C_\bullet$ is contractible. It turns out that this is true in the general case.

Exercise 2.5. *Show that a chain complex* $C_\bullet$ *is split exact if and only if it is contractible.*

Definition 2.6. Let $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ morphisms of chain complexes. Then $f_\bullet, g_\bullet$ are *chain homotopic* if their difference $f - g$ is null homotopic. In this case, the corresponding family of maps $\{s_n\}_{n\in\mathbb{Z}}$ is called a *chain homotopy*. We may define an equivalence relation on morphisms of chain complexes by identifying chain maps that are chain homotopic.

The map $H_n(f) : H_n(C) \to H_n(D)$ induced by a chain map $f_\bullet : C_\bullet \to D_\bullet$ depends only on the homotopy class of $f_\bullet$:

Proposition 2.7. *If* $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ *are homotopic maps of chain complexes, then these maps induce the same maps* $H_n(C_\bullet) \to H_n(D_\bullet)$ *on homology.*

Proof. It suffices to show that $H_n(f)$ is the zero map $H_n(C) \to H_n(D)$ for all $n \in \mathbb{Z}$ if $f : C_\bullet \to D_\bullet$ is null homotopic. Let $\{s_n\}_{n\in\mathbb{Z}}$ be a chain contraction of f, and let $[x] \in H_n(C)$, where $x \in Z_n(C)$. Then

$$f_n(x) = d^D_{n+1} s_n(x) + s_{n-1} d^C_n(x) = d^D_{n+1} s_n(x)$$

lies in $B_n(D)$, thus $[f(x)]$ is the zero element of $H_n(D)$. We conclude that $H_n(f)$ is the zero map $H_n(C) \to H_n(D)$. $\square$

Remark 2.8. *It can be shown that there is a homotopy category of chain complexes on $\mathcal{A}$, in which the objects are chain complexes on $\mathcal{A}$, and the morphisms are homotopy classes of chain morphisms. We have a special name for isomorphisms in this new category.*

Definition 2.9. A chain map $f_\bullet : C_\bullet \to D_\bullet$ defines a *chain homotopy equivalence* if it is an isomorphism in the homotopy category of chain complexes, i.e., if there is a chain map $g_\bullet : D_\bullet \to C_\bullet$ such that $f_\bullet g_\bullet$ is chain homotopic to the identity on $D_\bullet$ and $g_\bullet f_\bullet$ is chain homotopic to the identity on $C_\bullet$. If there exists a chain homotopy equivalence between two chain complexes, we say that they are *homotopy equivalent.*

3 Mapping Cones

The notion of homotopy in $\mathrm{Ch}(\mathcal{A})$ is closely related to the notion of homotopy in **Top**. Recall that continuous maps $f, g : X \to Y$ of topological spaces are homotopic in **Top** if there is a continuous map $H : [0,1] \times X \to Y$, called a homotopy, such that $H(0,x) = f(x)$ and $H(1,x) = g(x)$ for all $x \in X$. Equivalently, if $\iota_0 : X \to \{0\} \times X$ and $\iota_1 : X \to \{1\} \times X$ are the natural inclusions, then a continuous map $H : [0,1] \times X \to Y$ defines a homotopy between f and g if $H \circ \iota_0 = f$ and $H \circ \iota_1 = g$. Our aim is to find a counterpart for this definition in $\mathrm{Ch}(\mathcal{A})$. We first find a chain complex $I_\bullet$ to act as an interval object. We assume that $\mathcal{A} = R$—mod for a ring R.

Definition 3.1. Let $I_\bullet$ be the simplicial chain complex of an interval, consisting of two vertices v_0, v_1 and one edge $e = [v_0, v_1]$. That is, $I_0 = R\langle v_0, v_1\rangle$, $I_1 = R\langle e\rangle$, and $I_k = 0$ otherwise. Also, we have $\partial_1[v_0, v_1] = v_1 - v_0$ and $\partial_k = 0$ otherwise.

In **Top**, the domain of the homotopy H is the product of the interval $[0,1]$ with X. In $\mathrm{Ch}(\mathcal{A})$, the appropriate notion of a product is the tensor product.

Definition 3.2. Given chain complexes $C_\bullet, D_\bullet$, let $C_\bullet \otimes D_\bullet$ be the chain complex such that

$$(C_\bullet \otimes D_\bullet)_n = \bigoplus_{i+j=n} C_i \otimes D_j,$$

with the nth differential on $C_\bullet \otimes D_\bullet$ defined as follows: if i, j are integers such that $i + j = n$, and $(x, y) \in C_i \times D_j$, then

$$d_n(x \otimes y) = d_i^C x \otimes y + (-1)^i (x \otimes d_j^D y).$$

Remark 3.3. *Suppose that $C_\bullet$ is a chain complex. Then $I_\bullet \otimes C_\bullet$ is the chain complex such that at level n:*

$$\begin{aligned}(I_\bullet \otimes C_\bullet)_n &= (I_0 \otimes C_n) \oplus (I_1 \otimes C_{n-1}) \\ &= (\langle v_0\rangle \otimes C_n) \oplus (\langle v_1\rangle \otimes C_n) \oplus (\langle e\rangle \otimes C_{n-1}) \\ &= C_n^{v_0} \oplus C_n^{v_1} \oplus C_{n-1}^e,\end{aligned}$$

where $C_n^{v_i} = \langle v_i \rangle \otimes C_n$, and $C_{n-1}^e = \langle e \rangle \otimes C_{n-1}$. If $x \in C_n$, then

$$d_n(v_i \otimes x) = v_i \otimes d_n^C x,$$

and if $x \in C_{n-1}$, then

$$d_n(e \otimes x) = v_1 \otimes x - v_0 \otimes x - e \otimes d_{n-1}^C x.$$

There are also chain maps $\iota_0, \iota_1 : C_\bullet \to I_\bullet \otimes C_\bullet$ such that $\iota_0(x) = v_0 \otimes x$ and $\iota_1(x) = v_1 \otimes x$ for $x \in C_n$.

We give a new definition of homotopy in $\mathrm{Ch}(\mathcal{A})$ that mirrors the definition of homotopy in **Top** we saw earlier.

Definition 3.4. Chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are chain homotopic if there is a chain map $H : I_\bullet \otimes C_\bullet \to D_\bullet$ such that $H \circ \iota_0 = f_\bullet$ and $H \circ \iota_1 = g_\bullet$.

Proposition 3.5. *Chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are chain homotopic in the sense of Definition 2.6 if and only if they are chain homotopic in the sense of Definition 3.4.*

Proof. If $f_\bullet, g_\bullet$ are chain homotopic in the sense of Definition 3.4, then there is a chain map $H : I_\bullet \otimes C_\bullet \to D_\bullet$ such that $H \circ \iota_0 = f_\bullet$ and $H \circ \iota_1 = g_\bullet$. For $x \in C_n$, we have $e \otimes x \in (I \otimes C)_{n+1}$. Thus we may define $s_n : C_n \to D_{n+1}$ by letting $s_n(x) = H_{n+1}(e \otimes x)$. It follows that

$$\begin{aligned}
(d_{n+1}^D s_n + s_{n-1} d_n^C)(x) &= d_{n+1}^D H_{n+1}(e \otimes x) + H_n(e \otimes d_n^C x) \\
&\overset{!}{=} H_n d_{n+1}^{I \otimes C}(e \otimes x) + H_n(e \otimes d_n^C x) \\
&= H_n(v_1 \otimes x - v_0 \otimes x - e \otimes d_n^C x) + H_n(e \otimes d_n^C x) \\
&= H_n(v_1 \otimes x) - H_n(v_0 \otimes x) \\
&= H_n \circ \iota_1(x) - H_n \circ \iota_0(x) \\
&= g(x) - f(x),
\end{aligned}$$

where the marked equality follows because H is a chain map. Thus the maps $\{s_n\}_{n \in \mathbb{Z}}$ are a chain contraction of $f - g$, and we conclude that the chain maps $f_\bullet, g_\bullet$ are chain homotopic in the sense of Definition 2.6. Conversely, suppose that $\{s_n\}_{n \in \mathbb{Z}}$ is a chain contraction of $f - g$. Recall that $(I \otimes C)_n = C_n^{v_0} \oplus C_n^{v_1} \oplus C_{n-1}^e$. We define $H_n : (I \otimes C)_n \to D_n$ by requiring

$$H_n(v_0 \otimes x) = f_n(x) \quad \text{and} \quad H_n(v_1 \otimes x) = g_n(x)$$

for $x \in C_n$, and

$$H_n(e \otimes x) = s_{n-1}(x)$$

for $x \in C_{n-1}$. We leave it to the reader to show that $H : I_\bullet \otimes C_\bullet \to D_\bullet$ is a chain map such that $H \circ \iota_0 = f_\bullet$ and $H \circ \iota_1 = g_\bullet$, and thus that $f_\bullet, g_\bullet$ are chain homotopic in the sense of Definition 3.4. $\square$

Given a chain map $f_\bullet : C_\bullet \to D_\bullet$, we wish to define a new chain complex $\operatorname{cone}(f_\bullet)$, called the *mapping cone* of $f_\bullet$. This construction takes inspiration from the mapping cone C_f of a continuous map $f : X \to Y$ of topological spaces. Recall that to form C_f, we first take the cone CX of the space X (i.e., we take the quotient of the cylinder $I \times X$ by the equivalence relation that collapses $\{1\} \times X$ to a point) . Next, we glue CX to Y by taking the quotient of $CX \sqcup Y$ by the relation that glues $\{0\} \times X \subseteq CX$ to Y via $(0, x) \sim f(x)$. These two steps result in the mapping cone C_f. In $\mathrm{Ch}(\mathcal{A})$ we perform analogous maneuvers. First, we quotient the cylinder $I_\bullet \otimes C_\bullet$ by $\langle v_1 \rangle \otimes C_\bullet$. The object at the nth degree of this quotient is

$$\frac{(I_\bullet \otimes C_\bullet)_n}{\langle v_1 \rangle \otimes C_n} = \frac{C_n^{v_0} \oplus C_n^{v_1} \oplus C_{n-1}^e}{C_n^{v_1}} \cong C_n^{v_0} \oplus C_{n-1}^e.$$

The resulting chain complex is called the cone of $C_\bullet$ and is denoted by $\operatorname{cone}(C_\bullet)$ (cf. Definition 3.8). Next, we mimic the step of gluing CX to Y via f by taking the quotient of $\operatorname{cone}(C_\bullet) \oplus D_\bullet$ by a relation that identifies $C_n^{v_0}$ and D_n via $x \sim f_n(x)$ for $x \in C_n$. This results in a chain complex $\operatorname{cone}(f_\bullet)$, whose object at the nth degree is

$$\operatorname{cone}(f_\bullet)_n = C_{n-1} \oplus D_n.$$

The differentials for this chain complex are induced by the differential of the cylinder $I_\bullet \otimes C_\bullet$.

Definition 3.6. Given a chain morphism $f_\bullet : C_\bullet \to D_\bullet$, define the *mapping cone* of $f_\bullet$ to be the chain complex $\operatorname{cone}(f_\bullet)$ whose object in the nth degree is $C_{n-1} \oplus D_n$, and whose nth differential $C_{n-1} \oplus D_n \to C_{n-2} \oplus D_{n-1}$ satisfies

$$\begin{aligned} d_n(x, y) &= (-d_{n-1}^C(x), d_n^D(y) - f_{n-1}(x)) \\ &= \begin{bmatrix} -d_{n-1}^C & 0 \\ -f_{n-1} & d_n^D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

for $(x, y) \in C_{n-1} \oplus D_n$.

Remark 3.7. *We see that* $\operatorname{cone}(f_\bullet)$ *is indeed a chain complex by noting that the matrix representation of* $d_n d_{n+1}$ *is*

$$\begin{bmatrix} -d_{n-1}^C & 0 \\ -f_{n-1} & d_n^D \end{bmatrix} \begin{bmatrix} -d_n^C & 0 \\ -f_n & d_{n+1}^D \end{bmatrix} = \begin{bmatrix} d_{n-1}^C d_n^C & 0 \\ f_{n-1} d_n^C - d_n^D f_n & d_n^D d_{n+1}^D \end{bmatrix} = 0,$$

since $f_\bullet$ *is a chain map.*

Definition 3.8. Given a chain complex $C_\bullet$, define the cone of $C_\bullet$ to be the mapping cone of the identity chain map $\mathrm{id}_\bullet^C : C_\bullet \to C_\bullet$. The cone of $C_\bullet$ is denoted by $\operatorname{cone}(C_\bullet)$.

Proposition 3.9. *For any chain complex* $C_\bullet$*, the cone of* $C_\bullet$ *is split exact.*

Proof. We first find splitting maps $\{s_n : \mathrm{cone}(C_\bullet)_n \to \mathrm{cone}(C_\bullet)_{n+1}\}$. Let $s_n : C_{n-1} \oplus C_n \to C_n \oplus C_{n+1}$ be the map with matrix representation

$$s_n = \begin{bmatrix} 0 & -\mathrm{id}_n \\ 0 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} d_{n+1}s_n d_{n+1} &= \begin{bmatrix} -d_n & 0 \\ -\mathrm{id}_n & d_{n+1} \end{bmatrix} \begin{bmatrix} 0 & -\mathrm{id}_n \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -d_n & 0 \\ -\mathrm{id}_n & d_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & d_n \\ 0 & \mathrm{id}_n \end{bmatrix} \begin{bmatrix} -d_n & 0 \\ -\mathrm{id}_n & d_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} -d_n & 0 \\ -\mathrm{id}_n & d_{n+1} \end{bmatrix} \\ &= d_{n+1}, \end{aligned}$$

which confirms that $\mathrm{cone}(C_\bullet)$ is split. To see that $\mathrm{cone}(C_\bullet)$ is acyclic, note that for each n:

$$\begin{aligned} \ker(d_n) &= \ker \begin{bmatrix} -d_{n-1} & 0 \\ -\mathrm{id}_{n-1} & d_n \end{bmatrix} \\ &= \{(x, y) \in C_{n-1} \oplus C_n : d_{n-1}x = 0, x = d_n y\} \\ &= \{(d_n y, y) : y \in C_n\} \\ &= \mathrm{im} \begin{bmatrix} -d_n & 0 \\ -\mathrm{id}_n & d_{n+1} \end{bmatrix} \\ &= \mathrm{im}(d_{n+1}). \end{aligned}$$

This confirms that $\mathrm{cone}(C_\bullet)$ is split exact. □

Remark 3.10. *Proposition 3.9 is the homological realization of the fact that the cone CX of any topological space X is contractible.*

We demonstrate the value of mapping cones of chain maps in our final result, which reduces questions about quasi-isomorphisms to the study of exact complexes. We begin by recalling the definition of a quasi-isomorphism:

Definition 3.11. A chain map $f_\bullet : C_\bullet \to D_\bullet$ is called a *quasi-isomorphism* if its image $H_n(f_\bullet) : H_n(C_\bullet) \to H_n(D_\bullet)$ under the homology functor H_n is an isomorphism for every integer n.

Proposition 3.12. *A chain map $f_\bullet : C_\bullet \to D_\bullet$ is a quasi-isomorphism if and only if the mapping cone complex of $f_\bullet$ is exact.*

Proof. Given a chain map $f_\bullet : C_\bullet \to D_\bullet$, we claim that there is a short exact sequence of chain complexes

$$0 \longrightarrow D_\bullet \xrightarrow{\alpha_\bullet} \mathrm{cone}(f_\bullet) \xrightarrow{\beta_\bullet} C_\bullet[-1] \longrightarrow 0, \tag{10}$$

where $C_\bullet[-1]$ is the (-1)th translate of $C_\bullet$, the map $\alpha_\bullet$ satisfies $\alpha_n(x) = (0, x)$ for $x \in D_n$, and the map $\beta_\bullet$ satisfies $\beta_n(x, y) = -x$ for $(x, y) \in C_{n-1} \oplus D_n$. So as long as one remembers that the nth differential for $C_\bullet[-1]$ is $-d_n^C$ (cf. Translation 1.2.8 in Weibel's book), it is easy to show that $\alpha_\bullet, \beta_\bullet$ are chain maps, and that the sequence in (10) is exact. By Theorem 1.1, there is a long exact sequence

$$\cdots \to H_{n+1}(\mathrm{cone}_f) \to H_n(C) \xrightarrow{\partial_n} H_n(D) \to H_n(\mathrm{cone}_f) \to H_{n-1}(C) \to \cdots, \tag{11}$$

where we have recalled that $H_n(C[-1]) \cong H_{n-1}(C)$. Let $x \in Z_n(C)$ to compute the image of $[x] \in H_n(C)$ under the connecting homomorphism ∂_n. β_{n+1} maps $(-x, 0)$ to x. Next, we have

$$d_{n+1}^{\mathrm{cone}}(-x, 0) = \begin{bmatrix} -d_n^C & 0 \\ -f_n & d_{n+1}^D \end{bmatrix} \begin{bmatrix} -x \\ 0 \end{bmatrix} = \begin{bmatrix} d_n^C(x) \\ f_n(x) \end{bmatrix} = \begin{bmatrix} 0 \\ f_n(x) \end{bmatrix},$$

since $x \in Z_n(C)$. Lastly, we note that α_n maps $f_n(x)$ to $(0, f_n(x))$. Thus the image of $[x]$ under the connecting homomorphism ∂_n is

$$\partial_n[x] = [f_n(x)] = \tilde{f}_n[x].$$

In particular, the connecting homomorphism ∂_n is precisely the map $H_n(f) : H_n(C) \to H_n(D)$. Since the sequence in (11) is exact, we conclude that $f_\bullet : C_\bullet \to D_\bullet$ is a quasi-isomorphism if and only if $H_n(\mathrm{cone}_f) = 0$ for all n, that is, if and only if the mapping cone complex of $f_\bullet$ is exact. $\square$