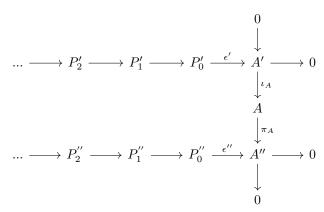
Homological algebra exercise sheet Week 5

1. Let \mathcal{S} be the category of short exact sequences

$$0 \to A \to B \to C \to 0 \tag{1}$$

in an abelian category \mathcal{A} and δ a homological δ -functor. Show that δ_i is a natural transformation from the functor sending (1) to $T_i(C)$ to the functor sending (1) to $T_{i-1}(A)$.

- 2. Show that a chain complex P is a projective object in \mathbf{Ch} if and only if it is a split exact complex of projectives. Hint : To see that P must be split exact, consider the surjection from $\mathit{cone}(id_p)$ to P[-1]. To see that split exact complexes are projective objects, consider the special case $0 \to P_1 \cong P_0 \to 0$.
- 3. Use the previous exercise to show that if \mathcal{A} has enough projectives, then so does the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes over \mathcal{A} .
- 4. Prove the Horseshoe Lemma. More concretely, given a commutaive diagram



where the column is exact and the rows are projective resolutions. Define $P_n = P'_n \oplus P''_n$. Prove P_{\bullet} assemble to form a projective resolution P of A, and the right-hand column lifts to an exact sequence of complexes

$$0 \to P'_{\bullet} \stackrel{\iota}{\to} P_{\bullet} \stackrel{\pi}{\to} P''_{\bullet} \to 0.$$

Here $\iota_n: P_n' \to P_n$ denotes the natural inclusion and $\pi_n: P_n \to P_n''$ denotes the natural projection.