Homologcal algebra exercise sheet Week 5

1. Let $\mathcal{S}$ be the category of short exact sequences
$$0 \to A \to B \to C \to 0 \tag{1}$$
in an abelian category $\mathcal{A}$ and δ a homological δ-functor. Show that δ_i is a natural transformation from the functor sending (1) to $T_i(C)$ to the functor sending (1) to $T_{i-1}(A)$.

2. Show that a chain complex P is a projective object in $\mathbf{Ch}$ if and only if it is a split exact complex of projectives. *Hint*: To see that P must be split exact, consider the surjection from $cone(id_p)$ to $P[-1]$. To see that split exact complexes are projective objects, consider the special case $0 \to P_1 \cong P_0 \to 0$.

3. Use the previous exercise to show that if $\mathcal{A}$ has enough projectives, then so does the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes over $\mathcal{A}$.

4. Prove the Horseshoe Lemma. More concretely, given a commutaive diagram
$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
... \arrow[r] & P'_2 \arrow[r] & P'_1 \arrow[r] & P'_0 \arrow[r, "\epsilon'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
 & & & & A \arrow[d, "\pi_A"] & \\
... \arrow[r] & P''_2 \arrow[r] & P''_1 \arrow[r] & P''_0 \arrow[r, "\epsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$
where the column is exact and the rows are projective resolutions. Define $P_n = P'_n \oplus P''_n$. Prove $P_\bullet$ assemble to form a projective resolution P of A, and the right-hand column lifts to an exact sequence of complexes
$$0 \to P'_\bullet \xrightarrow{\iota} P_\bullet \xrightarrow{\pi} P''_\bullet \to 0.$$
Here $\iota_n : P'_n \to P_n$ denotes the natural inclusion and $\pi_n : P_n \to P''_n$ denotes the natural projection.