Homological algebra exercise sheet Week 13

- 1. Give examples of maps f, g in $\mathbf{Ch}(\mathcal{A})$ such that
 - (i) f = 0 in $\mathbb{D}(A)$, but f is not null homotopic, and
 - (ii) g induces the zero map on cohomology, but $g \neq 0$ in $\mathbb{D}(A)$.

Hint: For (ii), try
$$X: 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0, Y: 0 \to \mathbb{Z} \xrightarrow{1} \mathbb{Z}/3 \to 0, g = (1,2).$$

- 2. (Proof of proposition 10.4.1.(2)). Let \mathcal{K} be a triangulated category and S a multiplicative system arising from a cohomological functor. We prove that $S^{-1}\mathcal{K}$ is triangulated. In the lectures,
 - We defined the translation T' by taking the translation $T: \mathcal{K} \to \mathcal{K}$ of \mathcal{K} and set T' as

$$T'(C) = T(C)$$
 for all $C \in Ob \mathcal{K}$

and on morphisms

$$T'(X \stackrel{s}{\leftarrow} X' \stackrel{f}{\rightarrow} Y) = T(X) \stackrel{T(s)}{\longleftarrow} T(X') \stackrel{T(f)}{\longrightarrow} T(Y).$$

(Which is well defined because S arises from a cohomological functor and by functoriality of T.)

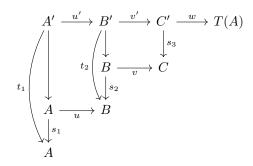
• We defined the exact triangles as follows. Consider fractions

$$A \stackrel{s_1}{\longleftarrow} A \xrightarrow{u} B$$

$$B \stackrel{s_2}{\longleftarrow} B \xrightarrow{v} C$$

$$C \stackrel{s_3}{\longleftarrow} C' \xrightarrow{w} T(A).$$

Composition is given by the Øre condition



so that

$$vs_2^{-1} \cong s_3 v't_2^{-1}$$
 and $us_1^{-1} \cong t_2 u't_1^{-1}$.

We say that $(us_1^{-1}, vs_2^{-1}, ws_3^{-1})$ is an exact triangle in $S^{-1}\mathcal{K}$ just in case (u', v', w) is an exact triangle in \mathcal{K} .

Verify that $S^{-1}\mathcal{K}$ satisfies the axioms TR1, TR2, TR3 of a triangulated category.

- 3. This is a follow up of Exercise 12.4. Let \mathcal{B} be a Serre subcategory of \mathcal{A} and let $\pi: \mathcal{A} \to \mathcal{A}/\mathcal{B}$ be the quotient map constructed in Exercise 12.4.
 - (a) Show that $H = \pi H^0 : \mathcal{K}(\mathcal{A}) \to \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is a cohomological functor, so that $\mathcal{K}_H(\mathcal{A})$ is a triangulated category. (See exercise 10.2.5 in Weibel's book).
 - (b) Show that X is in $\mathcal{K}_{\mathcal{B}}(\mathcal{A})$ iff the cohomology $H^{i}(X)$ is in \mathcal{B} for all i.
 - (c) Show that $\mathcal{K}_{\mathcal{B}}(\mathcal{A})$ is a localizing subcategory of $\mathcal{K}(\mathcal{A})$, and conclude that its localization $\mathbb{D}_{\mathcal{B}}(\mathcal{A})$ is a triangulated subcategory of $\mathbb{D}(\mathcal{A})$.
 - (d) Suppose that \mathcal{B} has enough injectives and that every injective object of \mathcal{B} is also injective in \mathcal{A} . Show that there is an equivalence

$$\mathbb{D}^+(\mathcal{B}) \cong \mathbb{D}^+_{\mathcal{B}}(\mathcal{A}).$$

4. Let R be a Noetherian ring, and let $\mathbb{M}(R)$ denote the category of all finitely generated R-modules. Let $\mathbb{D}_{fg}(R)$ denote the full subcategory of $\mathbb{D}(\mathbf{mod}-R)$ consisting of complexes A whose cohomology modules $H^i(A)$ are all finitely generated, that is, the category $\mathbb{D}_{\mathbb{M}(R)}(\mathbf{mod}-R)$ of exercise 13.3.

Show that $\mathbb{D}_{fg}(R)$ is a triangulated category and that there is an equivalence

$$\mathbb{D}^-(\mathbb{M}(R)) \cong \mathbb{D}^-_{fg}(\mathbb{M}(R)).$$

Hint. By exercise 12.4, $\mathbb{D}(R)$ *is a Serre subcategory of* \mathbf{mod} -R.

5. (Albelian derived category). The goal is to prove the following lemma.

Definition 0.1 (Semisimple abelian category). An abelian category is called semisimple if all short exact sequence split.

Lemma 0.2. Let A be an abelian category. The derived category $\mathbb{D}(A)$ is abelian if and only if A is semisimple.

(a) Show that if A is semisimple, for every morphism $f: A \to B$, there is a morphism $g: B \to A$ such that

$$fgf = f$$
 and $gfg = g$.

(b) Prove the following lemma.

Lemma 0.3 (Verdier). A triangulated category K is abelian if and only if every morphism $f: A \to B$ is isomorphic to

$$A' \oplus I \xrightarrow{\begin{pmatrix} 0 & 1_I \\ 0 & 0 \end{pmatrix}} I \oplus B'.$$

- (c) Using Lemma 0.3, prove that $\mathbb{D}(\mathcal{A})$ abelian implies that \mathcal{A} is semisimple.
- (d) Prove that if \mathcal{A} is semisimple, then $\mathbb{D}(\mathcal{A})$ is abelian.