Homological algebra exercise sheet Week 13

1. Give examples of maps f, g in $\mathbf{Ch}(\mathcal{A})$ such that

   (i) $f = 0$ in $\mathbb{D}(\mathcal{A})$, but f is not null homotopic, and

   (ii) g induces the zero map on cohomology, but $g \neq 0$ in $\mathbb{D}(\mathcal{A})$.

   Hint: For (ii), try $X : 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0, Y : 0 \to \mathbb{Z} \xrightarrow{1} \mathbb{Z}/3 \to 0, g = (1, 2)$.

2. *(Proof of proposition 10.4.1.(2)).* Let $\mathcal{K}$ be a triangulated category and S a multiplicative system arising from a cohomological functor. We prove that $S^{-1}\mathcal{K}$ is triangulated. In the lectures,

   - We defined the translation T' by taking the translation $T : \mathcal{K} \to \mathcal{K}$ of $\mathcal{K}$ and set T' as

     $$T'(C) = T(C) \text{ for all } C \in \operatorname{Ob}\mathcal{K}$$

     and on morphisms

     $$T'(X \xleftarrow{s} X' \xrightarrow{f} Y) = T(X) \xleftarrow{T(s)} T(X') \xrightarrow{T(f)} T(Y).$$

     (Which is well defined because S arises from a cohomological functor and by functoriality of T.)

   - We defined the exact triangles as follows. Consider fractions

     $$A \xleftarrow{s_1} A \xrightarrow{u} B$$
     $$B \xleftarrow{s_2} B \xrightarrow{v} C$$
     $$C \xleftarrow{s_3} C' \xrightarrow{w} T(A).$$

     Composition is given by the Øre condition

$$\begin{tikzcd}
A' \arrow[r, "u'"] \arrow[dd] \arrow[ddd, bend right, "t_1"'] & B' \arrow[r, "v'"] \arrow[d] \arrow[dd, bend right, "t_2"'] & C' \arrow[r, "w"] \arrow[d, "s_3"] & T(A) \\
 & B \arrow[r, "v"'] \arrow[d, "s_2"] & C & \\
A \arrow[r, "u"'] \arrow[d, "s_1"] & B & & \\
A & & &
\end{tikzcd}$$

so that
$$vs_2^{-1} \cong s_3 v' t_2^{-1} \text{ and } us_1^{-1} \cong t_2 u' t_1^{-1}.$$

We say that $(us_1^{-1}, vs_2^{-1}, ws_3^{-1})$ is an exact triangle in $S^{-1}\mathcal{K}$ just in case (u', v', w) is an exact triangle in $\mathcal{K}$.

Verify that $S^{-1}\mathcal{K}$ satisfies the axioms TR1, TR2, TR3 of a triangulated category.

3. This is a follow up of Exercise 12.4. Let $\mathcal{B}$ be a Serre subcategory of $\mathcal{A}$ and let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ be the quotient map constructed in Exercise 12.4.

   (a) Show that $H = \pi H^0 : \mathcal{K}(\mathcal{A}) \to \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is a cohomological functor, so that $\mathcal{K}_H(\mathcal{A})$ is a triangulated category. (See exercise 10.2.5 in Weibel's book).

   (b) Show that X is in $\mathcal{K}_\mathcal{B}(\mathcal{A})$ iff the cohomology $H^i(X)$ is in $\mathcal{B}$ for all i.

   (c) Show that $\mathcal{K}_\mathcal{B}(\mathcal{A})$ is a localizing subcategory of $\mathcal{K}(\mathcal{A})$, and conclude that its localization $\mathbb{D}_\mathcal{B}(\mathcal{A})$ is a triangulated subcategory of $\mathbb{D}(\mathcal{A})$.

   (d) Suppose that $\mathcal{B}$ has enough injectives and that every injective object of $\mathcal{B}$ is also injective in $\mathcal{A}$. Show that there is an equivalence
   $$\mathbb{D}^+(\mathcal{B}) \cong \mathbb{D}^+_\mathcal{B}(\mathcal{A}).$$

4. Let R be a Noetherian ring, and let $\mathbb{M}(R)$ denote the category of all finitely generated R-modules. Let $\mathbb{D}_{\mathrm{fg}}(R)$ denote the full subcategory of $\mathbb{D}(\mathbf{mod}-R)$ consisting of complexes A whose cohomology modules $H^i(A)$ are all finitely generated, that is, the category $\mathbb{D}_{\mathbb{M}(R)}(\mathbf{mod}-R)$ of exercise 13.3.

   Show that $\mathbb{D}_{\mathrm{fg}}(R)$ is a triangulated category and that there is an equivalence
   $$\mathbb{D}^-(\mathbb{M}(R)) \cong \mathbb{D}^-_{\mathrm{fg}}(\mathbb{M}(R)).$$

   Hint. By exercise 12.4, $\mathbb{D}(R)$ is a Serre subcategory of **mod**-R.

5. *(Albelian derived category).* The goal is to prove the following lemma.

   Definition 0.1 (Semisimple abelian category)**.** An abelian category is called semisimple if all short exact sequence split.

   Lemma 0.2. *Let $\mathcal{A}$ be an abelian category. The derived category $\mathbb{D}(\mathcal{A})$ is abelian if and only if $\mathcal{A}$ is semisimple.*

   (a) Show that if $\mathcal{A}$ is semisimple, for every morphism $f : A \to B$, there is a morphism $g : B \to A$ such that
   $$fgf = f \text{ and } gfg = g.$$

(b) Prove the following lemma.

Lemma 0.3 (Verdier)**.** *A triangulated category $\mathcal{K}$ is abelian if and only if every morphism $f : A \to B$ is isomorphic to*

$$A' \oplus I \xrightarrow{\begin{pmatrix} 0 & 1_I \\ 0 & 0 \end{pmatrix}} I \oplus B'.$$

(c) Using Lemma 0.3, prove that $\mathbb{D}(\mathcal{A})$ abelian implies that $\mathcal{A}$ is semisimple.

(d) Prove that if $\mathcal{A}$ is semisimple, then $\mathbb{D}(\mathcal{A})$ is abelian.