# Homological algebra exercise sheet Week 10

Throughout the sheet we fix abelian categories $\mathcal{A}$ and $\mathcal{B}$, assumed to have enough projectives. We let $A_\bullet \in \mathrm{Ch}(\mathcal{A})$, and fix a Cartan-Eilenberg resolution $P_{\bullet,\bullet} \to A_\bullet$ with augmentation $\varepsilon : P_{\bullet,0} \to A_\bullet$.

1. Recall that by definition we have that $B_p(\varepsilon) : B_p(P, d^h) \to B_p(A)$ and $H_p(\varepsilon) : B_p(P, d^h) \to H_p(A)$ are projective resolutions.

   (a) Show that $Z_p(\varepsilon) : Z_p(P, d^h) \to Z_p(A)$ and $\varepsilon_p : P_{p,\bullet} \to A_p$ are projective resolutions.

   (b) Suppose that $A_\bullet$ is bounded below. Show that $\varepsilon : \mathrm{Tot}^{\oplus}(P_{\bullet,\bullet}) \to A.$ is a quasi-isomorphism.

   (c) Show $(b)$ but now supposing that $\mathcal{A}$ satisfies axiom (AB4) (ie $\mathcal{A}$ is cocomplete and arbitrary direct sums of monics are monics). If you cannot show that $\varepsilon$ gives a quasi-isomorphism, show via a spectral sequence argument that $\mathrm{Tot}^{\oplus}(P_{\bullet,\bullet})$ and $A_\bullet$ have the same homologies.

2. Let $B_\bullet \in \mathrm{Ch}(\mathcal{A})$, and $f : A_\bullet \to B_\bullet$ be a chain map. If $Q_{\bullet,\bullet} \to B_\bullet$ is a Cartan-Eilenberg resolution, show that $f$ induces a map of double complexes $\tilde{f} : P_{\bullet,\bullet} \to Q_{\bullet,\bullet}$ (as mentioned in class, this map is unique up to homotopy).

3. Fix a right exact functor $F : \mathcal{A} \to \mathcal{B}$

   (a) Suppose $A_\bullet$ is concentrated in degree 0. Show that $\mathbb{L}_i(F(A)) = L_i F(A_0)$.

   (b) Let $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ be the subcategory of chain complexes in $\mathcal{A}$ trivial below degree 0, ie $A_\bullet \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ if $A_p = 0\ \forall p < 0$. Show that the hyper-left derived functors $\mathbb{L}_i F$, when restricted to $\mathrm{Ch}_{\geq 0}(\mathcal{A})$, are given by the ordinary derived functors of degree 0 homology, $L_i H_0 F$ (observe that $H_0 F$ is a right exact functor).

   (c) Show the dimension shifting formula $\mathbb{L}_i F(A[n]) = \mathbb{L}_{i+n} F(A)$

4. Let $A_\bullet$ be the mapping cone complex of $f$, $0 \longrightarrow A_1 \xrightarrow{f} A_0 \longrightarrow 0$. Show there is a long exact sequence

$$...\mathbb{L}_i F(A) \longrightarrow L_i F(A_1) \xrightarrow{L_i F(f)} L_i F(A_0) \longrightarrow \mathbb{L}_i F(A) \longrightarrow \mathbb{L}_{i-1} F(A) \longrightarrow ...$$

5. Let $f : X \to Y$ be a morphism of ringed spaces and $\mathcal{F}$ be an $\mathcal{O}_X$-module.

   (a) Suppose that $R^q f_* \mathcal{F} = 0$ for $q > 0$. Then show that $H^p(X, \mathcal{F}) = H^p(Y, f_* \mathcal{F})$ for all $p$.

   (b) Suppose that $H^p(Y, R^q f_* \mathcal{F}) = 0$ for all $q$ and $p > 0$. Then show that $H^q(X, \mathcal{F}) = H^0(Y, R^q f_* \mathcal{F})$ for all q.