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Solution 1

(a) In this case
$$k(\varphi) = \log \int_0^1 e^{\varphi y}\,\mathrm{d}y = \log\left(\frac{e^{\varphi}-1}{\varphi}\right), \quad \varphi \in \mathcal{N} = \mathbb{R},$$
with $k(0) = 0$ defined by continuity as $\varphi \to 0$. Hence the resulting exponential family is
$$f(y;\varphi) = e^{\varphi y - k(\varphi)} = \frac{\varphi e^{\varphi y}}{e^{\varphi}-1}, \quad y \in \mathcal{Y}, \varphi \in \mathbb{R}.$$

(b) In this case
$$k(\varphi) = \log \int_0^1 e^{\varphi_1 \log y + \varphi_2 \log(1-y)}\,\mathrm{d}y = \log \int_0^1 y^{\varphi_1}(1-y)^{\varphi_2}\,\mathrm{d}y$$
and we recognise this as a beta integral, defined for $\varphi_1, \varphi_2 > -1$, and then equal to $\Gamma(1+\varphi_1)\Gamma(1+\varphi_2)/\Gamma(2+\varphi_1+\varphi_2)$. The resulting density is therefore
$$f(y;\varphi) = \frac{\Gamma(2+\varphi_1+\varphi_2)}{\Gamma(1+\varphi_1)\Gamma(1+\varphi_2)} y^{\varphi_1}(1-y)^{\varphi_2}, \quad y \in \mathcal{Y}, \varphi = (\varphi_1,\varphi_2) \in (-1,\infty)^2,$$
which is just a rewriting of the beta density, for which $\alpha = \varphi_1 + 1$ and $\beta = \varphi_2 + 1$, and $\alpha, \beta > 0$, in the usual notation.

Solution 2 Using the results on linear combinations of normal variables, $\overline{X} \stackrel{\mathrm{D}}{=} \mu + \sigma n^{-1/2} Z$, where $Z \sim \mathcal{N}(0,1)$. Hence $Y \stackrel{\mathrm{D}}{=} 1/(\mu + \sigma n^{-1/2} Z)$.

If we apply the delta method with $g(u) = 1/u$, we have $g'(u) = -1/u^2$, provided $u \neq 0$. If $\mu \neq 0$ and as $n \to \infty$, therefore,
$$Y = g(\overline{X}) \stackrel{\mathrm{D}}{=} g(\mu + \sigma n^{-1/2} Z) \doteq g(\mu) + \sigma n^{-1/2} Z g'(\mu) \mathrel{\dot\sim} \mathcal{N}\left\{g(\mu), g'(\mu)^2 \times \sigma^2/n\right\} = \mathcal{N}\{1/\mu, \sigma^2/(n\mu^4)\}.$$
Note that if X has units of length (say), then its mean and its variance have units of length and length^2, so $1/X$ has units of 1/length and its variance has units of 1/length^2, agreeing with the distribution here.

If $\mu = 0$, then for any n,
$$\mathrm{P}(\overline{X} < 0) = \mathrm{P}(\overline{X} > 0) = \mathrm{P}(\mu + \sigma n^{-1/2} Z > 0) = \mathrm{P}(Z > 0) = 1/2,$$
and if $y > 0$ we can write
$$\begin{aligned} \mathrm{P}(Y > y \mid \overline{X} > 0) &= \mathrm{P}\{1/(\mu + \sigma n^{-1/2} Z) > y \mid Z > 0\} \\ &= \mathrm{P}\{Z < n^{1/2}/(\sigma y) \mid Z > 0\} \\ &= 2\left[\Phi\left\{n^{1/2}/(y\sigma)\right\} - 1/2\right] \\ &\to 1, \quad n \to \infty, \end{aligned}$$
so as this is true for any positive y, the distribution of Y will concentrate at $\pm\infty$ with equal probabilities.

Solution 3

(a) Clearly $\tau = 0$ is one possible solution for any $R_0 > 0$. The slopes of $1 - \tau$ and $e^{-R_0\tau}$ at $\tau = 0$ are respectively -1 and $-R_0$, and it is clear from a plot of these two functions against τ that there is a second, positive, solution to the equation within the interval $\tau \in (0,1)$ if $R_0 > 1$. The intuition is that if $R_0 \leq 1$, then the epidemic is certain to be of negligible size, but that if $R_0 > 1$ then it may be of negligible size (if by chance it dies out immediately) but otherwise will affect a positive fraction of the population.

(b) The equation $1-\tau = e^{-R_0\tau}$ gives $R_0 = -\tau^{-1}\log(1-\tau) = g(\tau)$, say, which is a smooth function of τ for $\tau \in (0,1)$. Hence the delta method applies, and $\widehat{R}_0 = g(\widehat{\tau})$ is asymptotically normal with mean $g(\tau) = R_0$ and with variance given using

$$g'(\tau) = \log(1-\tau)/\tau^2 + 1/\{\tau(1-\tau)\} = -R_0/\tau + 1/\{\tau(1-\tau)\} = \{1 - R_0(1-\tau)\}/\{\tau(1-\tau)\};$$

the delta method variance formula $\sigma_R^2 = g'(\tau)^2\sigma^2$ gives the required result.

(c) The variance of a constant is zero, so if the constant itself is non-zero, $c^2 = 0/T = 0$. If $T \sim U(0,d)$ then $\mathrm{E}(T) = d/2$ and $\mathrm{var}(T) = d^2/12$, leading to $c^2 = 1/3$. An exponential variable with mean μ has variance μ^2, giving $c^2 = 1$. The exponential distribution has high coefficient of variation (for example, the gamma distribution has $c = \alpha^{-1/2}$, where α is the shape parameter, and taking $\alpha > 1$ gives a unimodal distribution that seems potentially suitable for T), so taking $c = 1$ should give an upper bound for σ^2 and hence for σ_R^2.

Solution 4

(a) The density is

$$f(y) = \frac{\alpha\theta^\alpha}{(\theta+y)^{\alpha+1}} = \exp\left\{-(\alpha+1)\log(\theta+y) + \log\alpha + \alpha\log\theta\right\}, \quad y > 0, \quad \alpha, \theta > 0.$$

This is not of exponential family form, because no term in the log density can be written as $s(y)^{\mathrm{T}}\varphi(\theta)$ for some function $s(y)$ that depends only on y.

(b) The method-of-moments estimator satisfies $\overline{Y} = \tilde{\theta}/(\alpha-1)$, i.e., $\tilde{\theta} = (\alpha-1)\overline{Y}$, and this is easily checked to be unbiased using the given formula for $\mathrm{E}(Y)$ and the fact that $\mathrm{E}(\overline{Y}) = \mathrm{E}(Y)$.

Moreover $\mathrm{var}(\overline{Y}) = \mathrm{var}(Y)/n$, so

$$\mathrm{var}(\tilde{\theta}) = (\alpha-1)^2\mathrm{var}(\overline{Y}) = \frac{(\alpha-1)^2}{n}\mathrm{var}(Y) = \frac{\theta^2\alpha}{n(\alpha-2)}.$$

(c) The bias of of $\tilde{\theta}_c$ is

$$b(\tilde{\theta}_c;\theta) = \mathrm{E}(\tilde{\theta}_c) - \theta = c\frac{\theta}{\alpha-1} - \theta = \frac{(\alpha-1-c)\theta}{\alpha-1},$$

and $\mathrm{var}(\tilde{\theta}_c) = c^2\mathrm{var}(\overline{Y})$, so (after a little algebra) the mean squared error of $\tilde{\theta}_c$ is

$$b(\tilde{\theta}_c;\theta)^2 + c^2\mathrm{var}(\overline{Y}) = \frac{\theta^2}{n(\alpha-1)^2(\alpha-2)}\left\{n(\alpha-2)(\alpha-1-c)^2 + c^2\alpha\right\}.$$

This is minimised by differentiating the expression $\{\cdot\}$ here with respect to c, giving

$$\frac{\mathrm{d}\{\cdot\}}{\mathrm{d}c} = -2n(\alpha-2)(\alpha-1-c) + 2c\alpha = 0,$$

which results in

$$c = \frac{n(\alpha-1)(\alpha-2)}{n(\alpha-2)+\alpha} \to \alpha - 1, \quad n \to \infty.$$

Note that for any finite n the estimator is biased, but it is asymptotically unbiased, and that the second derivative $\mathrm{d}^2\{\cdot\}/\mathrm{d}c^2 = 2n(\alpha-1)(\alpha-2) + 2\alpha$ is positive — which is obvious because the MSE is a sum of positive quadratic expressions in c.