Solution 1

(a) The most powerful test against any fixed value of $\mu \neq \mu_0$ is obtained from the Neyman–Pearson lemma. The likelihood ratio for testing $\mu = \mu_0$ against $\mu = \mu_1$ with σ known is

$$\begin{aligned}\frac{f_1(y_1,\ldots,y_n)}{f_0(y_1,\ldots,y_n)} &= \frac{(2\pi\sigma^2)^{-n/2}\exp\{-\frac{1}{2\sigma^2}\sum_{j=1}^n(y_j-\mu_1)^2\}}{(2\pi\sigma^2)^{-n/2}\exp\{-\frac{1}{2\sigma^2}\sum_{j=1}^n(y_j-\mu_0)^2\}}\\ &= \exp\left[\frac{1}{2\sigma^2}\left\{2n\overline{y}(\mu_1-\mu_0)-\mu_1^2+\mu_0^2\right\}\right].\end{aligned}$$

This is monotone increasing in $\overline{y}$ for any fixed $\mu_1 > \mu_0$, and so the critical region rejects H_0 when $\overline{y} \geq t_\alpha$, with t_α chosen to give a test of size α. The null distribution of $\overline{Y}$ is $\mathcal{N}(\mu_0, \sigma^2/n)$, so

$$\alpha = \mathrm{P}_0(\overline{Y} \geq t_\alpha) = \mathrm{P}_0\left\{n^{1/2}(\overline{Y}-\mu_0)/\sigma \geq n^{1/2}(t_\alpha-\mu_0)/\sigma\right\} = 1-\Phi\left\{n^{1/2}(t_\alpha-\mu_0)/\sigma\right\},$$

which implies that $n^{1/2}(t_\alpha - \mu_0)/\sigma = z_{1-\alpha}$, giving $t_\alpha = \mu_0 + \sigma n^{-1/2} z_{1-\alpha}$ and thus $\mathcal{Y}_\alpha^+$, as required. When $\mu_1 < \mu_0$, a similar computation leads to

$$\mathcal{Y}_\alpha^- = \left\{(y_1,\ldots,y_n) : \overline{y} \leq \mu_0 + \sigma n^{-1/2} z_\alpha\right\}.$$

(b) The critical region $\mathcal{Y}_\alpha^+$ is most powerful for any $\mu_1 > \mu_0$, so it is uniformly most powerful for $\mu_1 > \mu_0$, and likewise for $\mathcal{Y}_\alpha^-$ against the alternatives $\mu < \mu_0$.

(c) Symmetry of the distribution of $\overline{Y} - \mu_0$ under the null hypothesis implies that $\mathcal{Y}_\beta$ has size

$$\mathrm{P}_0(Y \in \mathcal{Y}_\beta) = \mathrm{P}_0\left(n^{1/2}|\overline{Y}-\mu_0|/\sigma \geq z_{1-\beta}\right) = 2\mathrm{P}_0\left\{n^{1/2}(\overline{Y}-\mu_0)/\sigma \geq z_{1-\beta}\right\} = 2\beta,$$

so we should choose $\beta = \alpha/2$ to achieve size α. $\mathcal{Y}_{\alpha/2}$ is not uniformly most powerful of size α, because if $\mu_1 > \mu_0$ then $\mathcal{Y}_\alpha^+$ also has size α but has higher power (because $z_{1-\alpha} \leq z_{1-\alpha/2}$).

Solution 2

(a) As $\min(Y_1,\ldots,Y_r) > x$ if and only if $Y_1 > x, \ldots, Y_r > x$, we have

$$\mathrm{P}\{\min(Y_1,\ldots,Y_r) \leq x\} = 1-\mathrm{P}\{\min(Y_1,\ldots,Y_r) > x\} = 1-\mathrm{P}(Y_1 > x)^r = 1-\exp(-r\lambda x), \quad x > 0,$$

and for $x, y > 0$, $\mathrm{P}(Y - x > y \mid Y > x)$ equals

$$\frac{\mathrm{P}(Y-x>y, Y>x)}{\mathrm{P}(Y>x)} = \frac{\mathrm{P}(Y>y+x)}{\mathrm{P}(Y>x)} = \exp\{-\lambda(x+y)\}/\exp(-\lambda x) = \exp(-\lambda y),$$

as required.

(b) As $\mathrm{P}(E_j/\lambda \leq x) = \mathrm{P}(E_j \leq \lambda x) = 1-\exp(-\lambda x) = \mathrm{P}(Y_j \leq x)$, we have $Y_j \stackrel{\mathrm{D}}{=} E_j/\lambda$. We argue as follows:

- $Y_{(1)}$ is the smallest of n independent exponential variables, so it is exponential with parameter $n\lambda$ and therefore we can write $Y_{(1)} \stackrel{\mathrm{D}}{=} E_1/(n\lambda)$;
- the remaining $n-1$ variables have the lack of memory property, so given that $Y_{(1)} = x$ the remaining $Y_j - x$ have exponential distributions with parameter λ. Thus $Y_{(2)} - Y_{(1)}$ is the minimum of $n-1$ exponential variables, i.e., $Y_{(2)} - Y_{(1)} \stackrel{\mathrm{D}}{=} E_2/\{(n-1)\lambda\}$;
- iterating the argument by successively conditioning on $Y_{(2)}, \ldots, Y_{(n-1)}$ and obtaining the distributions of $Y_{(3)} - Y_{(2)}, \ldots, Y_{(n)} - Y_{(n-1)}$ gives the stated representation.

(c) A standard exponential variable has mean and variance both equal to 1, so

$$\mathrm{E}(Y_{(r)}) = \frac{1}{\lambda}\sum_{j=1}^{r}\frac{1}{n+1-j}, \quad \mathrm{cov}(Y_{(r)}, Y_{(s)}) = \frac{1}{\lambda^2}\sum_{j=1}^{m}\frac{1}{(n+1-j)^2}, \quad r, s, \in \{1, \dots, n\},$$

with $m = \min(s, r)$ and the second formula giving the variance when $r = s$. Note the simple approximate integral formulae

$$\begin{aligned}\mathrm{E}(Y_{(r)}) &\doteq \frac{1}{\lambda}\int_{n-r+\frac{1}{2}}^{n+\frac{1}{2}}\frac{\mathrm{d}x}{x} = \lambda^{-1}\log\{(n+\tfrac{1}{2})/(n-r+\tfrac{1}{2})\}, \\ \mathrm{cov}(Y_{(r)}, Y_{(s)}) &\doteq \frac{1}{\lambda^2}\int_{n-m+\frac{1}{2}}^{n+\frac{1}{2}}\frac{\mathrm{d}x}{x^2} = \lambda^{-2}\frac{m}{(n+\frac{1}{2})(n-m+\frac{1}{2})}.\end{aligned}$$

Solution 3

(a) Clearly if P is small then $-\log P$ is large, and

$$\mathrm{P}(-\log P \le x) = \mathrm{P}(P \ge e^{-x}) = 1 - e^{-x}, \quad x > 0,$$

so $-\log P$ has a standard exponential distribution. Thus S_F, a sum of independent exponential variables, has a gamma distribution, with upper tail probability

$$\mathrm{P}_0(S_F \le s) = \int_0^s \frac{x^{n-1}}{n!}e^{-x}\,\mathrm{d}x,$$

and quantiles s_α, say. The critical region is $\{(p_1, \dots, p_n) \in (0,1)^n : -\sum_{j=1}^n \log p_j \ge s_{1-\alpha}\}$.

(b) Here $\mathrm{P}_0(S_T > s) = \mathrm{P}(P_1 > s, \dots, P_n > s) = (1-s)^n$ for $s \in (0,1)$, and the critical region is $\{(p_1, \dots, p_n) \in (0,1)^n : \min_j p_j \le 1 - (1-\alpha)^{1/n}\}$.

(c) Under this alternative we have

$$\mathrm{P}(-\log P \le x) = \mathrm{P}(P \ge e^{-x}) = 1 - (e^{-x})^{1/\gamma} = 1 - e^{-x/\gamma},$$

so $-\log P \sim \exp(1/\gamma)$ with $\gamma > 1$. This is an exponential family and we are comparing the simple null and alternative hypotheses $\gamma = 1$ and $\gamma > 1$, so Example 30 of the notes applies. The likelihood ratio for $p_1, \dots, p_n$ is

$$\frac{f_1(p)}{f_0(p)} = \frac{\gamma^{-n}\prod_{j=1}^n p_j^{1/\gamma-1}}{\prod_{j=1}^n 1} = \exp\left\{-\sum_{j=1}^n \log p_j(1-1/\gamma) - n\log\gamma\right\},$$

which is an exponential family with $\varphi = -1/\gamma < 0$, $s^* = -\sum_j \log p_j$, $k(\varphi) = n\log\gamma = -n\log(-\varphi)$ and $\log m^*(p) = -\sum_j \log p_j$. Since φ is a monotone increasing function of γ, the computation in the example implies that the most powerful test has a critical region of the form $s^* > s_{1-\alpha}$, and therefore S_F is the best test statistic in this situation. As we always have $\gamma > 1$ or equivalently $\varphi < -1$ under the alternative, it is also uniformly most powerful.

(d) This extends (c), with the log likelihood ratio turning out to be

$$(a-1)\sum \log p_j + (b-1)\sum \log(1-p_j) = (b-a)\{wS_F + (1-w)S_P\}.$$

(e) In this situation the cumulative distribution function for P is $(1-q)x + qx^{1/\gamma}$, so the density is $(1-q) + (q/\gamma)x^{1/\gamma-1}$, for $x \in (0,1)$. As $\gamma > 1$, this implies that that the density is unbounded as $x \to 0$, which may not be so plausible, but in any case the obvious approach would be to estimate q and γ (for example using maximum likelihood) and hence decide whether $q = 0$ or $\gamma > 1$. In this case S_T seems attractive, because it seems likely that it would be able to profit from the spike under the alternative.