Problem 1 Let $Y_1, \ldots, Y_n \stackrel{\text{iid}}{\sim} U(a, b)$. Find minimal sufficient statistics for θ when (a) $a = -\theta$, $b = \theta$, and (b) $a = \theta - 1$, $b = \theta + 1$. If the minimal sufficient statistic is not scalar, can you find an ancillary?

Problem 2 Find minimal sufficient statistics in the following settings:

(a) $Y_1, \ldots, Y_N \stackrel{\text{iid}}{\sim} \text{Poiss}(\theta)$ with N a geometric random variable with success probability θ;

(b) $Y_1, \ldots, Y_n \stackrel{\text{ind}}{\sim} \text{Poiss}(\theta_1, \ldots, \theta_n)$ for fixed n, where $\log \theta_j = x_j^{\mathrm{T}} \beta$ depends on known $d \times 1$ vectors of covariates $x_1, \ldots, x_n$ and an unknown parameter $\beta \in \mathbb{R}^d$; and

(c) $Y_1, \ldots, Y_n \stackrel{\text{iid}}{\sim} \exp(\lambda)$. In this last case show also that $(Y_1/\overline{Y}, \ldots, Y_n/\overline{Y})$ is distribution-constant, and without computing their joint density show that it is independent of $\overline{Y}$.

Problem 3 If Y_1/θ and $Y_2\theta$ are independent gamma variables with unit scale parameter and shape parameter n, check that their joint density function is

$$f(y_1, y_2; \theta) = \frac{(y_1 y_2)^{n-1}}{\Gamma(n)^2} \exp(-y_1/\theta - \theta y_2), \quad y_1, y_2 > 0, \theta > 0.$$

(a) Show that this is a $(2, 1)$ exponential family with minimal sufficient statistic $S = (T, A)$, where $T = (Y_1/Y_2)^{1/2}$ and $A = (Y_1 Y_2)^{1/2}$.

(b) Find the joint density of T and A, show that A is ancillary, and find the conditional density of T given A.

(c) Show that the observed information for θ is proportional to a, compute the unconditional Fisher information, and hence discuss the role of A.

Problem 4 Independent exponential random variables Y_1 and Y_2 have respective densities $\theta_1 e^{-\theta_1 y_1}$ and $\theta_2 e^{-\theta_2 y_2}$, where $\theta_1, \theta_2 > 0$, and $\lambda, \psi > 0$ below.

(a) Find the joint density of Y_1 and Y_2 when $\theta_1 = \lambda$ and $\theta_2 = \lambda + \psi$. Inspect this and hence eliminate λ and thus obtain a $1 - 2\alpha$ confidence interval for ψ.

(b) If $\theta_1 = \lambda$ and $\theta_2 = \lambda\psi$ show that $\psi Y_2/Y_1$ is a pivot and find a $1 - 2\alpha$ confidence interval for ψ.

(c) If $\theta_1 = \lambda$ and $\theta_2 = \lambda\psi$, show that λ can be eliminated by conditioning on $W_\psi = Y_1 + \psi Y_2$, and that the conditional distribution of $T = Y_1$ given W_ψ is

$$\mathrm{P}(T \le t \mid W_\psi = w_\psi; \psi) = \frac{t}{w_\psi}, \quad 0 < t < w_\psi.$$

Deduce that the resulting confidence interval for ψ is the same as in (b).