Problem 1

(a) I draw a sample of two balls at random without replacement from a bag containing two red balls and three white balls. Give the sample space for this random experiment. Are its elements equiprobable? What is the probability that both balls in the sample are red, given that at least one is red?

(b) If $\mathrm{P}(X > x) = 1/x^2$, for $x > 1$, find the probability density function of $Y = 1/X$.

(c) Find the median of $Y = \exp(X)$, where $X \sim U(a, b)$ for $a < b$.

(d) If $X \sim \exp(\lambda)$, find the distribution and density functions of $Y = \cos X$.

(e) A simple model for a daily rainfall amount X is that $X = 0$ with probability $1-p$ and otherwise is exponential with parameter λ. Find the probability that $X = 0$ given that $X < 3$, and obtain the mean and variance of X.

(f) The joint probability mass function of random variables (X, Y) is given by the table:

	x		
y	1	3	5
2	c	$2c$	$3c$
4	$3c$	$2c$	c

Find $\mathrm{E}(X)$ and $\mathrm{E}(X \mid Y = 4)$. Are X and Y independent?

(g) If $Z_j \stackrel{\text{ind}}{\sim} \mathcal{N}(a_j, 1)$ $(j = 1, \ldots, n)$ independent of $Y \sim \mathcal{N}(\mu, \sigma^2)$, find the joint distribution of $X_j = Y + Z_j$ and say under what circumstances they are finitely exchangeable.

(h) If $X_1, \ldots, X_N \stackrel{\text{iid}}{\sim} \text{Poiss}(\lambda)$, $\mathrm{P}(N = n) = p(1-p)^{n-1}$ for $n \in \{1, 2 \ldots\}$, and $I(\cdot)$ is the indicator function, find the mean and variance of $T = \sum_{j=1}^{N} I(X_j = 0)$. How would the mean of T change if $X_1, \ldots, X_N$ were dependent?

(i) If $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} \text{Pois}(\lambda)$, find the approximate distribution of $Y = 2\overline{X}^{1/2}$ for large n.

(j) If $(X_1, Y_n), \ldots, (X_n, Y_n)$ form a random sample from a bivariate distribution with means and (finite positive) variances (μ_X, μ_Y) and (σ_X^2, σ_Y^2) and correlation ρ, with $\mu_X \neq 0$, show that $T = \sum_j Y_j / \sum_j X_j$ converges in probability to $\theta = \mu_Y/\mu_X$ and that for large n

$$n^{1/2}\mu_X(T - \theta) \mathrel{\dot\sim} \mathcal{N}(0, \sigma_X^2\theta^2 - 2\theta\rho\sigma_X\sigma_Y + \sigma_Y^2).$$

How would you estimate the variance?

(k) Two successive software downloads take times (minutes) $X_1 \sim \mathcal{N}(8, 3^2)$ and $X_2 \sim \mathcal{N}(16, 4^2)$. The download times are independent. Find (i) the distribution of the total download time $T = X_1 + X_2$, (ii) the probability that T exceeds 30 minutes, (iii) the probability that T exceeds 30 minutes, given that $X_1 = 10$, and (iv) the probability that X_1 was less than 7 minutes, given that $T = 30$ minutes.

(l) Three friends arrive to dine together independently at random between 7 and 8 o'clock. Give the density functions of the times of the first and last arrivals, and the expected time between them.