Problem 1 A random sample $y_1, \ldots, y_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$ with average $\overline{y}$ is to be used to test the null hypothesis $H_0 : \mu = \mu_0$ against the alternative $\mu = \mu_1$; below σ^2 is known and $z_p = \Phi^{-1}(p)$.

(a) Show that if $\mu_1 > \mu_0$ then the most powerful critical region of size α is

$$\mathcal{Y}_\alpha^+ = \left\{ y \in \mathbb{R}^n : \overline{y} \geq \mu_0 + \sigma n^{-1/2} z_{1-\alpha} \right\},$$

and find the corresponding most powerful critical region $\mathcal{Y}_\alpha^-$ when $\mu_1 < \mu_0$.

(b) Are $\mathcal{Y}_\alpha^+$ and $\mathcal{Y}_\alpha^-$ uniformly most powerful against their respective alternatives? Explain.

(c) Now consider the two-sided alternative $H : \mu_1 \neq \mu_0$. Compute the size of the critical region

$$\mathcal{Y}_\beta = \left\{ y \in \mathbb{R}^n : n^{1/2} |\overline{y} - \mu_0| / \sigma \geq z_{1-\beta} \right\}$$

and hence give a two-sided critical region of size α. Is this uniformly most powerful against H?

Problem 2 Consider the order statistics $0 < Y_{(1)} < \cdots < Y_{(n)}$ of a random sample $Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} \exp(\lambda)$.

(a) Show that $\min(Y_1, \ldots, Y_r) \sim \exp(r\lambda)$, and that each Y_j has the lack-of-memory property

$$\mathrm{P}(Y - x > y \mid Y > x) = \mathrm{P}(Y > y), \quad x, y > 0.$$

(b) Show that $Y_j \overset{\mathrm{D}}{=} E_j / \lambda$ with $E_1, \ldots, E_n \overset{\text{iid}}{\sim} \exp(1)$, and hence obtain the *Renyi representation*

$$Y_{(r)} \overset{\mathrm{D}}{=} \frac{1}{\lambda} \sum_{j=1}^{r} \frac{E_j}{n + 1 - j}, \quad r = 1, \ldots, n.$$

(c) Find the means and covariances of $Y_{(1)}, \ldots, Y_{(n)}$.

Problem 3 Below we consider different ways to combine evidence from independent P-values $P_1, \ldots, P_n$ from testing the same null hypothesis.

(a) Find the distributions of $-\log P_j$ and hence of $S_F = -\sum_j \log P_j$ (Fisher's statistic) and $S_P = -\sum_j \log(1 - P_j)$ (Pearson's statistic). Give the size α critical regions for tests based on S_F and S_P.

(b) Give the size α critical region for a test based on $S_T = \min_j P_j$ (Tippett's statistic).

(c) Suppose that the alternative is such that $\mathrm{P}(P \leq x) = x^{1/\gamma}$ for $x \in (0, 1)$ and some $\gamma > 1$. Which of S_F, S_P and S_T is preferable, and why?

(d) If P has density $x^{a-1}(1 - x)^{b-1} / B(a, b)$, where $0 < x < 1$, $0 < a < 1$, $b \geq 1$ and $a \neq b$, show that the uniformly most powerful test involves $wS_F + (1 - w)S_P$, where $w = (1 - a)/(b - a)$.

(e) What would you do if it is believed that the null hypothesis holds in a proportion $1 - q$ of the tests and the alternative in (c) holds in the remaining ones, with both q and γ unknown?