

Problem 1 Let $Y_1, \dots, Y_n$ be a random sample from the gamma density $\lambda^\alpha y^{\alpha-1} e^{-\lambda y} / \Gamma(\alpha)$, for $y > 0$ and $\alpha, \lambda > 0$. Find the expected information matrix $\imath(\alpha, \lambda)$, in terms of the digamma function $\Psi(\alpha) = d \log \Gamma(\alpha) / d\alpha$ its derivative Ψ' (the trigamma function); note that $\Psi(1) \doteq -0.577$ and $\Psi'(1) \doteq 1.645..$

- (a) Find the score statistic for testing whether $\alpha = 1$ when λ is known, and give its large-sample distribution.
- (b) Find the score statistic for testing whether $\alpha = 1$ when λ is unknown, and give its large-sample distribution.
- (c) Show that the parameter $\mu = \alpha/\lambda$ is orthogonal to α , (i) by computing $\imath(\alpha, \mu)$, and (ii) using the fact that the gamma density is an exponential family. Find the score statistic corresponding to (b) in this orthogonal parametrisation. Comment.

Problem 2

In an exponential family $m(y) \exp\{s\theta - k(\theta)\}$ with scalar θ , $q(\theta)$ equals the Wald statistic $t(\theta) = \hat{\jmath}^{1/2}(\hat{\theta} - \theta)$.

Compute the elements of $r^*(\theta)$ for random samples from the (a) $\mathcal{N}(\theta, 1)$ and (b) $\mathcal{N}(0, 1/\theta)$ distributions, and comment on the quality of the approximations.

Hint for (b): check the examples in the course.

Problem 3

- (a) Show that the modified likelihood root is invariant when $\varphi(\theta)$ is replaced by $a + B\varphi(\theta)$, where $a_{d \times 1}$ and $B_{d \times d}$ are constants and B is non-singular.
- (b) Show that in the location-scale example with normal $h(u)$, we can set $\varphi(\theta) = (\eta/\tau^2, 1/\tau^2)^\top$.

Problem 4

If Y_1 and Y_2 are independent exponential variables with means γ^{-1} and $(\gamma\psi)^{-1}$, show that a parameter $\lambda(\psi, \gamma)$ orthogonal to ψ solves the equation $\partial\gamma/\partial\psi = -\gamma/(2\psi)$, and (without solving this PDE, unless you feel the urge) verify that a possible solution is $\lambda = \gamma\psi^{1/2}$.